

Rate of convergence estimates for the zero dissipation limit in Abelian sandpiles

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May 11, 2011

Abstract: We consider a continuous height version of the Abelian sandpile model with small amount of bulk dissipation $\gamma > 0$ on each toppling, in dimensions $d = 2, 3$. In the limit $\gamma \downarrow 0$, we give a power law upper bound, based on coupling, on the rate at which the stationary measure converges to the discrete critical sandpile measure. The proofs are based on a coding of the stationary measure by weighted spanning trees, and an analysis of the latter via Wilson's algorithm. In the course of the proof, we prove an estimate on coupling a geometrically killed loop-erased random walk to an unkilld loop-erased random walk.

Key-words: Abelian sandpile, weighted spanning trees, Wilson's algorithm, zero-dissipation limit, self-organized criticality.

1 Introduction

In the paper [7] a continuous height version of the Abelian sandpile model was studied. The reason for interest in that model is that it allows an arbitrarily small amount of dissipation on every toppling, and hence yields a natural family of subcritical models approximating the discrete (critical) sandpile. In the present paper we study the dissipative models on \mathbb{Z}^d , $d = 2, 3$, and give a power law upper bound on the rate at which the stationary measure of the dissipative model converges to the critical sandpile measure. Our proof also applies in $d = 4$, but would only yield a logarithmic bound. Our methods break down for $d \geq 5$. Hence it remains an open problem to give a power law bound in dimensions $d \geq 4$.

Let us first recall the definition of the discrete Abelian sandpile model. See the survey by Redig [16] for general background, and the paper [5] for a nice introduction to the basic facts. Let $\Lambda \subset \mathbb{Z}^d$ be finite. We define the set of stable configurations in Λ as

$$\Omega_{\Lambda}^{\text{discr}} = \{0, 1, \dots, 2d - 1\}^{\Lambda}.$$

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We also consider the set of all non-negative configurations $\mathcal{X}_\Lambda^{\text{discr}} = \{0, 1, \dots\}^\Lambda$. Consider the *toppling matrix*

$$\Delta_{xy} = \begin{cases} 2d & \text{if } x = y; \\ -1 & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases} \quad x, y \in \Lambda,$$

where $x \sim y$ denotes that x and y are adjacent on the \mathbb{Z}^d lattice. Let $\eta \in \mathcal{X}_\Lambda^{\text{discr}}$. If $\eta_x \geq 2d$, we say that x can be *legally toppled*, and the result of toppling x is the new configuration $T^{(x)}\eta$ defined by

$$(T^{(x)}\eta)_y := \eta_y - \Delta_{xy}, \quad y \in \Lambda.$$

Note that if the toppling was legal, $T^{(x)}\eta \in \mathcal{X}_\Lambda^{\text{discr}}$. If a finite sequence of legal topplings has a stable result, it is called the *stabilization of η* , and is denoted $\mathcal{S}_\Lambda^{\text{discr}}(\eta)$. It is well-known [3, 16, 5] that stabilization is well-defined as a map $\mathcal{S}_\Lambda^{\text{discr}} : \mathcal{X}_\Lambda^{\text{discr}} \rightarrow \Omega_\Lambda^{\text{discr}}$.

The dynamics of the model is defined as a Markov chain $(\eta(n))_{n \geq 0}$ with state space $\Omega_\Lambda^{\text{discr}}$. Let X_1, X_2, \dots be i.i.d. with a fixed distribution $\{p(x)\}_{x \in \Lambda}$ on Λ , where $p(x) > 0$, $x \in \Lambda$. At time $n \geq 1$ a new particle is added at X_n , and the configuration is stabilized. That is, $\eta(n) := \mathcal{S}_\Lambda^{\text{discr}}(\eta(n-1) + \delta_{X_n, \cdot})$, where δ_{xy} is Kronecker's delta. It is well-known [3, 16] (see also [5, Corollary 2.16]) that the Markov chain has a single recurrent class $\mathcal{R}_\Lambda^{\text{discr}}$, and the stationary distribution, denoted ν_Λ , is uniform on $\mathcal{R}_\Lambda^{\text{discr}}$ (irrespective of the choice of p).

We now give the definition of the continuous height dissipative model in finite volume $\Lambda \subset \mathbb{Z}^d$. Let $\gamma \geq 0$ be a real parameter, and define the set of stable configurations $\Omega_\Lambda^{(\gamma)} := [0, 2d + \gamma)^\Lambda$. Let us also define the set of all non-negative configurations $\mathcal{X}_\Lambda := [0, \infty)^\Lambda$. Consider the toppling matrix given by:

$$\Delta_{xy}^{(\gamma)} = \begin{cases} 2d + \gamma & \text{if } x = y; \\ -1 & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases} \quad x, y \in \Lambda.$$

Let $\eta \in \mathcal{X}_\Lambda$. If $\eta_x \geq 2d + \gamma$ for some $x \in \Lambda$, then we say that x can be *γ -legally toppled*, and the result of toppling x is the new configuration $T_x^{(\gamma)}\eta$ defined by $(T_x^{(\gamma)}\eta)_y := \eta_y - \Delta_{xy}^{(\gamma)}$, $y \in \Lambda$. If a finite sequence of γ -legal topplings has a stable result, it is called the *γ -stabilization of η* , and is denoted $\mathcal{S}_\Lambda^{(\gamma)}(\eta)$. By arguments similar to the discrete case, it is not difficult to show that $\mathcal{S}_\Lambda^{(\gamma)}$ is well-defined as a map $\mathcal{S}_\Lambda^{(\gamma)} : \mathcal{X}_\Lambda \rightarrow \Omega_\Lambda^{(\gamma)}$. see for example [3] or [4, Appendix B].

The dynamics of the dissipative model is defined as a Markov chain $(\eta(n))_{n \geq 0}$ with state space $\Omega_\Lambda^{(\gamma)}$. Let X_1, X_2, \dots be i.i.d. with distribution p . At time $n \geq 1$, unit height is added at X_n and the configuration is stabilized according to the toppling matrix $\Delta^{(\gamma)}$, that is, $\eta(n) = \mathcal{S}_\Lambda^{(\gamma)}(\eta(n-1) + \delta_{X_n, \cdot})$. Similarly to the discrete model,

a set of recurrent configurations $\mathcal{R}_\Lambda^{(\gamma)}$ can be defined, and Lebesgue measure on $\mathcal{R}_\Lambda^{(\gamma)}$ is invariant for the dynamics [7, Section 2.2]. We denote the invariant probability measure by $m_\Lambda^{(\gamma)}$.

It was shown in [1, Theorem 1] and [6, Appendix] that for any $d \geq 2$, the measures ν_Λ weakly converge to a limit ν on the space $\Omega^{\text{discr}} := \{0, 1, \dots, 2d-1\}^{\mathbb{Z}^d}$, as $\Lambda \uparrow \mathbb{Z}^d$ (the limit can be taken along any sequence of Λ 's that exhaust \mathbb{Z}^d). It is quite clear that the measure $m_\Lambda^{(0)}$ should be closely related to ν_Λ . Indeed, based on results further explained in Section 2, one also easily gets that $m_\Lambda^{(0)}$ has a unique weak limit $m^{(0)}$.

It was further shown in [7, Lemma 4], that for all $d \geq 2$ and any $\gamma > 0$, the weak limit $m_\Lambda^{(\gamma)} \Rightarrow m^{(\gamma)}$ exists for all $d \geq 2$. It was also shown in [7, Proposition 4] that as $\gamma \downarrow 0$, $m^{(\gamma)} \Rightarrow m^{(0)}$. The goal of the present paper is to prove the following theorem. Let $\Omega^{(\gamma)} := [0, 2d + \gamma)^{\mathbb{Z}^d}$ and let $\Omega := [0, 2d)^{\mathbb{Z}^d}$.

Theorem 1. *Suppose that the cylinder event $E \subset \Omega$ depends only on the heights in $B(k) = [-k, k]^d \cap \mathbb{Z}^d$.*

(1) *If $d = 3$, there exist constants $C < \infty$, $\eta > 0$ such that for all $\gamma < 1$*

$$\left| m^{(\gamma)}(E) - m^{(0)}(E) \right| \leq Ck^2\gamma^\eta + Ck^5(\log k)\gamma.$$

(2) *If $d = 2$, there exist constants $c_0 > 0$ and $C, C_0 < \infty$ such that for all $\gamma \leq c_0k^{-C_0}$ we have*

$$\left| m^{(\gamma)}(E) - m^{(0)}(E) \right| \leq Ck^{21/23}\gamma^{1/46-o(1)}, \quad (1)$$

with $o(1)$ denoting a positive quantity that approaches 0 as $\gamma \rightarrow 0$.

Remark 1. For certain special events, a better exponent of γ was obtained in [7, Proposition 5]. There the estimate for $d \geq 3$ is of the form $C(k)\gamma$ and for $d = 2$ of the form $C(k)\gamma \log(1/\gamma)$. A natural question is whether these represent the precise rate of convergence for all events.

Remark 2. A suitable value of C_0 is determined from the proof. For $\gamma \leq c(2k)^{-23}$ we have the somewhat worse bound $Ck^{3/2}\gamma^{1/46-o(1)}$.

2 Discretized heights and spanning trees

Our proof of Theorem 1 is based on a certain “discretization” of the measure $m^{(\gamma)}$, and a coding of the discretized measure by weighted spanning trees. These tools were already introduced in [7], however, here we will need somewhat more detailed properties of the coding than in [7]. The form of the coding follows from the proof of [1, Theorem 1]. The coding is the main link between Theorem 1 and the loop-erased random walk estimates that make up most of the proof. We explain the coding in detail in this Section.

2.1 Allowed configurations

Let $\Lambda \subset \mathbb{Z}^d$ be finite. The sets $\mathcal{R}_\Lambda^{\text{discr}}$ and $\mathcal{R}_\Lambda^{(\gamma)}$ admit the following description. Let $\eta \in \Omega_\Lambda^{\text{discr}}$ or $\eta \in \Omega_\Lambda^{(\gamma)}$ with some $\gamma \geq 0$. For $\emptyset \neq W \subset \Lambda$ write η_W for the restriction of the configuration η to W . We say that η_W is a *forbidden subconfiguration (FSC)*, if

$$\eta_y < |\{z \in W : z \sim y\}|, \quad y \in W,$$

where $|A|$ denotes the cardinality of the set A . A configuration $\eta \in \Omega_\Lambda^{\text{discr}}$ or $\eta \in \Omega_\Lambda^{(\gamma)}$ is called *allowed*, if there is no $\emptyset \neq W \subset \Lambda$ such that η_W is a FSC. It was shown in [3] that all elements of $\mathcal{R}_\Lambda^{\text{discr}}$ are allowed, and it was shown in [13], using the “Burning Test” explained below, that all allowed configurations in $\Omega_\Lambda^{\text{discr}}$ are in $\mathcal{R}_\Lambda^{\text{discr}}$. The analogous statement is also true in the continuous case: it follows from [4, Appendix E] that $\mathcal{R}_\Lambda^{(\gamma)}$ consists precisely of the allowed configurations of $\Omega_\Lambda^{(\gamma)}$.

2.2 Discretization and the uniformity property

The characterization in terms of allowed configurations shows that $m_\Lambda^{(\gamma)}$ can be completely understood in terms of a discrete measure. Let

$$\Omega_\Lambda^{\text{discr},*} := \{0, 1, \dots, 2d-1, 2d\}^\Lambda,$$

and define the map $\psi_\Lambda : \mathcal{X}_\Lambda \rightarrow \Omega_\Lambda^{\text{discr},*}$ via

$$\xi_x := (\psi_\Lambda(\eta))_x = \begin{cases} h & \text{if } h \leq \eta_x < h+1 \text{ for some } h \in \{0, 1, 2, \dots, 2d-1\}; \\ 2d & \text{if } \eta_x \geq 2d. \end{cases} \quad (2)$$

We can also view ψ_Λ as a map from $\Omega_\Lambda^{(\gamma)}$ in a natural way, for any $\gamma \geq 0$. It follows directly from the definition of FSCs that for any $\gamma \geq 0$ and any $\eta \in \Omega_\Lambda^{(\gamma)}$, we have the equivalence:

$$\eta \text{ is allowed} \quad \Longleftrightarrow \quad \xi = \psi_\Lambda(\eta) \text{ is allowed.}$$

This together with the fact that $m_\Lambda^{(\gamma)}$ is normalized Lebesgues measure, implies the following statement.

$$\begin{aligned} &\text{Under the measure } m_\Lambda^{(\gamma)}, \text{ and given the value of } \xi = \psi_\Lambda(\eta), \text{ the variables } (\eta_x)_{x \in \Lambda} \text{ are conditionally independent with conditional distributions: } \eta_x \sim \text{Unif}(h, h+1) \text{ when } \xi_x = h, \ h = 0, 1, \dots, 2d-1, \text{ and} \\ &\eta_x \sim \text{Unif}(2d, 2d+\gamma) \text{ when } \xi_x = 2d. \end{aligned} \quad (3)$$

We will denote by $\nu_\Lambda^{(\gamma)}$ the image of $m_\Lambda^{(\gamma)}$ under the map ψ_Λ . By the definition of ψ_Λ , the measure $\nu_\Lambda^{(0)}$ concentrates on $\Omega_\Lambda^{\text{discr}}$, and in fact coincides with ν_Λ , by the characterization in terms of allowed configurations.

We now proceed to describe ν_Λ and $\nu_\Lambda^{(\gamma)}$ when $\gamma > 0$. Let us write

$$\mathcal{A}_\Lambda = \{\eta \in \Omega_\Lambda^{\text{discr},*} : \eta \text{ is allowed}\}.$$

We first observe that $\nu_\Lambda^{(\gamma)}$ obeys a certain weighting depending on γ . For $\xi \in \mathcal{A}_\Lambda$, let $H(\xi) = |\{x \in \Lambda : \xi_x = 2d\}|$. Since $m_\Lambda^{(\gamma)}$ is normalized Lebesgue measure, (3) implies that

$$\nu_\Lambda^{(\gamma)}(\xi) = c\gamma^{H(\xi)}, \quad \xi \in \mathcal{A}_\Lambda, \quad (4)$$

for some constant $c = c(\gamma, \Lambda)$. On the other hand, as stated before, ν_Λ is uniform on $\mathcal{A}_\Lambda \cap \Omega_\Lambda^{\text{discr}}$.

2.3 The Burning Test

There is a simple algorithm, the Buring Test [3], that checks if a configuration is allowed or not. There is some flexibility in setting up the algorithm. We choose one here that will work for both $\xi \in \Omega_\Lambda^{\text{discr}}$ and $\xi \in \Omega_\Lambda^{\text{discr},*}$. Set $U_0 := \Lambda$. We call U_0 the set of vertices *unburnt at time 0*. We define

$$\begin{aligned} B_1 &:= \{x \in U_0 : \xi_x = 2d\} \\ U_1 &:= \Lambda \setminus B_1. \end{aligned}$$

We inductively define for $i \geq 2$:

$$\begin{aligned} B_i &:= \{x \in U_{i-1} : \eta_x \geq |\{y \in U_{i-1} : y \sim x\}|\}; \\ U_i &:= U_{i-1} \setminus B_i. \end{aligned}$$

We call B_i and U_i the set of vertices *burning at time i* and *unburnt at time i* , respectively. We say that the algorithm *terminates*, if $U_i = \emptyset$ for some i . Using the definition of FSCs, it can be shown by induction on i that there cannot be any FSC containing a vertex in B_i , $i \geq 1$, and therefore, if the algorithm terminates, then η is allowed. On the other hand, if the algorithm does not terminate, and $\emptyset \neq U_i = U_{i+1} = \dots =: U$, then η_U is an FSC. Hence the algorithm terminates if and only if η was allowed.

2.4 The Majumdar-Dhar bijection with spanning trees

A useful description of ν_Λ and $\nu_\Lambda^{(\gamma)}$ can be given in terms of weighted spanning trees, that we now describe. This bijection was discovered by Majumdar and Dhar [13], and the extension given here to $\nu_\Lambda^{(\gamma)}$ is from [7].

We now define the multigraph $G_\Lambda = (V(G_\Lambda), E(G_\Lambda))$ that will carry the weighted spanning trees. We first define an infinite graph G . Add a new vertex ϖ to \mathbb{Z}^d , so the vertex set of G is $\mathbb{Z}^d \cup \{\varpi\}$. The edge set of G consists of: (i) for each $x, y \in \mathbb{Z}^d$ that are adjacent in the \mathbb{Z}^d lattice, we place an edge between x and y ; (ii) for each $x \in \mathbb{Z}^d$ we place an edge between x and ϖ . We call the edges of type (i) *ordinary*,

and we call those of type (ii) *dissipative*. It will be convenient to refer to the ordinary edge between x and y as $\text{ord}(x, y) = \text{ord}(y, x)$, and the dissipative edge between x and ϖ as $\text{diss}(x)$. For later use, we also define the graph $G^{(0)}$, that is obtained by removing ϖ and all dissipative edges from G (in other words, the usual \mathbb{Z}^d lattice). The multigraph G_Λ is now defined by identifying all vertices of G in the set $\mathbb{Z}^d \setminus \Lambda$ with ϖ , and removing loops. In G_Λ we still call edges dissipative or ordinary, according to their origin. We also use the notation $\text{ord}(x, y)$ and $\text{diss}(x)$ according to the origin of the edge. In particular, if $x \in \Lambda$, $y \in \mathbb{Z}^d \setminus \Lambda$ and x and y are adjacent in \mathbb{Z}^d , then $\text{ord}(x, y)$ denotes an ordinary edge of G_Λ between x and ϖ . The usefulness of this is that there can be more than one ordinary edges between x and ϖ , and these are thereby distinguished. We further define the graph $G_\Lambda^{(0)}$ by removing all dissipative edges from G_Λ (but keeping the vertex ϖ).

Let \mathcal{T}_Λ denote the set of spanning trees of G_Λ , and let $\mathcal{T}_\Lambda^{(0)}$ denote those spanning trees that do not contain dissipative edges. The latter is naturally identified with the set of spanning trees of $G_\Lambda^{(0)}$. We now define a map $\sigma_\Lambda : \mathcal{A}_\Lambda \rightarrow \mathcal{T}_\Lambda$. Let $\xi \in \mathcal{A}_\Lambda$. We define the spanning tree $t = \sigma_\Lambda(\xi)$ in stages. At each stage, we will connect each vertex in B_i to some vertex in B_{i-1} , which automatically ensures that there are no loops. In order to be able to start, we define $B_0 = \{\varpi\}$. Since $\cup_{i \geq 0} B_i = \Lambda \cup \{\varpi\} = V(G_\Lambda)$, the construction ensures that we get a spanning tree.

First we connect each $x \in B_1 = H(\xi)$ to ϖ , that is we put the edge $\text{diss}(x)$ into t . Suppose that $i \geq 2$, and each vertex in $\cup_{1 \leq j < i} B_j$ has been connected to an earlier vertex. Let $x \in B_i$, and define

$$\begin{aligned} n_{x,\Lambda} &:= \text{number of ordinary edges between } x \text{ and } \cup_{0 \leq j < i} B_j; \\ P_{x,\Lambda} &:= \{y \in \mathbb{Z}^d : |y| = 1 \text{ and } \text{ord}(x, x+y) \text{ is an edge between } x \text{ and } B_{i-1}\}; \\ K_{x,\Lambda} &:= \{2d - n_x, \dots, 2d - n_x + |P_{x,\Lambda}| - 1\}. \end{aligned} \quad (5)$$

Suppose that for every set $\emptyset \neq P \subset \{y \in \mathbb{Z}^d : |y| = 1\}$ and every set $K \subset \{0, 1, \dots, 2d - 1\}$ of the form $\{k, k+1, \dots, k+|P|-1\}$ an arbitrary bijection $\alpha_{P,K} : P \rightarrow K$ is fixed. The fact that x burns at time i means that we must have $\xi_x \in K_{x,\Lambda}$: x has $2d - n_x$ unburnt neighbours at time i , so in order for it to burn at time i , we must have $\xi_x \geq 2d - n_x$. On the other hand, since it did not burn before time i , we must have $\xi_x \leq 2d - n_x + |P_{x,\Lambda}| - 1$. We select the edge $\text{ord}(x, x + \alpha_{P_{x,\Lambda}, K_{x,\Lambda}}^{-1}(\xi_x))$ between x and B_{i-1} to be placed in t . This completes the definition of $t = \sigma_\Lambda(\xi)$.

Lemma 1 (Majumdar-Dhar [13], [7]).

- (i) The map σ_Λ is a bijection between \mathcal{A}_Λ and \mathcal{T}_Λ .
- (ii) The restriction of σ_Λ to $\mathcal{A}_\Lambda \cap \Omega_\Lambda^{\text{discr}}$ is a bijection between this set and $\mathcal{T}_\Lambda^{(0)}$.

Proof. (i) We show that σ_Λ is injective. Let $\xi^1, \xi^2 \in \mathcal{A}_\Lambda$, $\xi^1 \neq \xi^2$, and let $t^1 := \sigma_\Lambda(\xi^1)$, $t^2 := \sigma_\Lambda(\xi^2)$. If $H(\xi^1) = B_1(\xi^1) \neq B_1(\xi^2) = H(\xi^2)$, then t^1 and t^2 differ in at least one dissipative edge. Hence we may assume that $B_1(\xi^1) = B_1(\xi^2)$, and this implies that $\xi^1 = \xi^2$ on $B_1(\xi^1) = B_1(\xi^2)$. Let $i \geq 2$ be the smallest index such that either

$B_i(\xi^1) \neq B_i(\xi^2)$ or there exists $x \in B_i(\xi^1) = B_i(\xi^2)$ with $\xi_x^1 \neq \xi_x^2$. If such index did not exist, we would get by induction on i that $\xi^1 = \xi^2$ on $\cup_{i \geq 1} B_i(\xi^1) = \cup_{i \geq 1} B_i(\xi^2) = \Lambda$, a contradiction. By the choice of i , we have

$$B_j(\xi^1) = B_j(\xi^2) \text{ for } 1 \leq j \leq i-1. \quad (6)$$

If $B_i(\xi^1) \neq B_i(\xi^2)$, then pick a vertex x in the symmetric difference. Then by the construction of σ_Λ , in one of t^1 and t^2 there is an edge from x to $B_{i-1}(\xi^1) = B_{i-1}(\xi^2)$ and there is no such edge in the other, so $t^1 \neq t^2$. Suppose therefore that $B_i(\xi^1) = B_i(\xi^2)$, but there exists $x \in B_i(\xi^1) = B_i(\xi^2)$ such that $\xi_x^1 \neq \xi_x^2$. By the equality (6), we have $n_{x,\Lambda}(\xi^1) = n_{x,\Lambda}(\xi^2)$, $P_{x,\Lambda}(\xi^1) = P_{x,\Lambda}(\xi^2)$, and hence also $K_{x,\Lambda}(\xi^1) = K_{x,\Lambda}(\xi^2)$. However, since $\xi_x^1 \neq \xi_x^2$ we have $\alpha_{P_{x,\Lambda}, K_{x,\Lambda}}^{-1}(\xi_x^1) \neq \alpha_{P_{x,\Lambda}, K_{x,\Lambda}}^{-1}(\xi_x^2)$, and therefore the edge between x and B_{i-1} is different in t^1 and t^2 . This completes the proof of injectivity.

We now show that σ_Λ is surjective. In the course of doing so, we find the inverse map $\sigma_\Lambda^{-1} =: \varphi_\Lambda : \mathcal{T}_\Lambda \rightarrow \mathcal{A}_\Lambda$. First we note that for any $\xi \in \mathcal{A}_\Lambda$, the sets B_0, B_1, \dots and the data in (5) can be easily expressed in terms of $t = \sigma_\Lambda(\xi)$ as well. Namely, let $d_t(\cdot, \cdot)$ denote graph distance in the tree t . Then due to the construction of t , we have

$$\begin{aligned} B_0 &= \{\varpi\}; \\ B_1 &= \{x \in \Lambda : \text{diss}(x) \in t\}; \\ B_i &= \{x \in \Lambda : d_t(B_0 \cup B_1, x) = i-1\}, \quad i \geq 2. \end{aligned} \quad (7)$$

Since this expresses B_0, B_1, \dots in terms of t , the formulas (5) show that $n_{x,\Lambda}$, $P_{x,\Lambda}$ and $K_{x,\Lambda}$ are also expressed in terms of t . Also, by the definition of σ_Λ , if the unique edge of t in $P_{x,\Lambda}$ is $\text{ord}(x, x+y)$, then we have $\xi_x = \alpha_{P_{x,\Lambda}, K_{x,\Lambda}}(y)$.

The above makes it clear what the inverse $\varphi_\Lambda = \sigma_\Lambda^{-1}$ has to be. Suppose that $t \in \mathcal{T}_\Lambda$ is given. We use (7) to *define* the B_i 's and for $x \in B_i$, $i \geq 2$, we use (5) as the definition of $n_{x,\Lambda}$, $P_{x,\Lambda}$ and $K_{x,\Lambda}$. For $x \in B_1$, we set $\xi_x = 2d$. For $x \in B_i$, $i \geq 2$ let $y_{x,\Lambda} \in \mathbb{Z}^d$ be such that $\text{ord}(x, x+y_{x,\Lambda})$ is the unique edge of t in $P_{x,\Lambda}$, and we set $\xi_x = \alpha_{P_{x,\Lambda}, K_{x,\Lambda}}(y_{x,\Lambda})$. We define $\varphi_\Lambda(t) := \xi$. It is clear that if $\xi \in \mathcal{A}_\Lambda$, then $\sigma_\Lambda(\varphi_\Lambda(t)) = t$. What is left to show is that we always have $\xi \in \mathcal{A}_\Lambda$.

We prove that for every $t \in \mathcal{T}_\Lambda$ we have $\xi = \varphi_\Lambda(t) \in \mathcal{A}_\Lambda$, by applying the Burning Test to ξ . It is immediate that in the first step exactly B_1 burns as $B_1 = H(\xi)$ by construction. Suppose now inductively that $i \geq 2$ and we already know that at time $1 \leq j \leq i-1$ exactly B_j burns. Let $x \in B_i$. Then due to the inductive hypothesis and the definition of $n_{x,\Lambda}$, x has precisely $2d - n_{x,\Lambda}$ unburnt neighbours at time $i-1$. Since $\xi_x \in K_{x,\Lambda}$ by the definition of ξ , we have $\xi_x \geq 2d - n_{x,\Lambda}$ and hence x burns at time i . Let now $x \in B_j$ with $j \geq i+1$. Then by the induction hypothesis, B_{j-1}, B_j, \dots are unburnt at time $i-1$, and hence the number of unburnt neighbours of x at time $i-1$ is at least $2d - n_{x,\Lambda} + |P_{x,\Lambda}|$. Hence, since $\xi_x \in K_{x,\Lambda}$, we have $\xi_x < 2d - n_x + |P_{x,\Lambda}|$, and therefore x does not burn at time i . This shows that at time i precisely the set B_i burns, and completes the induction. Therefore ξ is allowed, and we have shown that σ_Λ is a bijection between \mathcal{A}_Λ and \mathcal{T}_Λ .

(ii) This second statement of the Lemma is now clear from the construction. \square

For $\gamma \geq 0$, let $\mu_\Lambda^{(\gamma)}$ be the probability measure on \mathcal{T}_Λ such that $\mu_\Lambda^{(\gamma)}(t) = c\gamma^{H(t)}$, where $H(t)$ = number of dissipative edges in t . In particular, $\mu_\Lambda = \mu_\Lambda^{(0)}$ is the uniform spanning tree measure on $G_\Lambda^{(0)}$. It is clear from (4) that for all $\gamma \geq 0$ the measure $\nu_\Lambda^{(\gamma)}$ is precisely the image of $\mu_\Lambda^{(\gamma)}$ under $\varphi_\Lambda = \sigma_\Lambda^{-1}$.

In Sections 2.5, 2.6 below, we describe the limits of the measures $\mu_\Lambda^{(\gamma)}$, $\nu_\Lambda^{(\gamma)}$, $m_\Lambda^{(\gamma)}$, $\gamma \geq 0$, as $\Lambda \uparrow \mathbb{Z}^d$.

2.5 Infinite volume limits when $\gamma = 0$

When $\gamma = 0$, it is well known [15] that the weak limit $\mu = \mu^{(0)} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_\Lambda^{(0)}$ exists, it is called the Wired Uniform Spanning Forest measure [2]. The paper [1] proves that the weak limit $\nu = \nu^{(0)} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu_\Lambda^{(0)}$ also exists, for any $d \geq 2$. In the cases $d = 2, 3$ relevant for this paper, it follows from the proof in [1] that the coding in terms of spanning trees remains true in \mathbb{Z}^d , in the following sense. Recall that the graph $G^{(0)}$ is the \mathbb{Z}^d lattice, and let $\tilde{\Omega}^{(0)} := \{0, 1\}^{E(G^{(0)})}$. Also recall that $\Omega^{\text{discr}} = \{0, 1, \dots, 2d - 1\}^{\mathbb{Z}^d}$. There exists a measurable map $\varphi^{(0)} : \tilde{\Omega}^{(0)} \rightarrow \Omega^{\text{discr}}$ defined $\mu^{(0)}$ -a.e. such that the image of $\mu^{(0)}$ under φ is $\nu^{(0)}$.

We now state the form of $\varphi^{(0)}$ in the relevant cases $d = 2, 3$. This is similar to the form of φ_Λ in Section 2.4. First note that $\mu^{(0)}$ concentrates on configurations in $\tilde{\Omega}^{(0)}$ that are trees with one end [15, Theorem 4.3]. (We say that a tree has one end, if any two infinite self-avoiding paths in the tree have infinitely many vertices in common.) Fix $\omega \in \tilde{\Omega}^{(0)}$ that is a tree with one end, and write $d_\omega(\cdot, \cdot)$ for graph distance in the tree ω . For any $x \in \mathbb{Z}^d$, let $\pi_x^{(0)}$ denote the unique self-avoiding path in ω from x to infinity. There exists a unique vertex V_x that is on all the paths $\{\pi_{x+y}^{(0)}\}_{|y| \leq 1}$ and is furthest from infinity (along each path). We define

$$\begin{aligned} n_x^{(0)} &= \left| \{y \in \mathbb{Z}^d : |y| = 1, d_\omega(x + y, V_x) < d_\omega(x, V_x)\} \right| \\ P_x^{(0)} &:= \left\{ y \in \mathbb{Z}^d : |y| = 1, d_\omega(x + y, V_x) = d_\omega(x, V_x) - 1 \right\} \\ K_x^{(0)} &:= \{2d - n_x^{(0)}, \dots, 2d - n_x^{(0)} + |P_x^{(0)}| - 1\} \\ y_x^{(0)} &:= \text{the vertex such that } (x, x + y_x^{(0)}) \text{ is the first edge of } \pi_x^{(0)}. \end{aligned} \tag{8}$$

Set $\xi_x := \alpha_{P_x^{(0)}, K_x^{(0)}}(y_x^{(0)})$, $x \in \mathbb{Z}^d$. It follows from the proof of [1, Theorem 1] and the form of φ_Λ in Section 2.4 that $\xi =: \varphi^{(0)}(\omega)$ is the claimed map. In other words, $\{\xi_x\}_{x \in \mathbb{Z}^d}$ has distribution $\nu^{(0)}$, if ω has distribution $\mu^{(0)}$.

Taking into account the uniformity property (3), it is fairly straightforward to show [7, Lemma 4] that we also have the weak limit $m^{(0)} := \lim_{\Lambda \uparrow \mathbb{Z}^d} m_\Lambda^{(0)}$. Also, the uniformity property is preserved in this limit.

2.6 Infinite volume limits when $\gamma > 0$

We now describe what happens for $\gamma > 0$. Due to the monotonicity properties of weighted spanning tree measures [2, Sections 4,5] it follows that the weak limit $\mu^{(\gamma)} := \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_\Lambda^{(\gamma)}$ exists for any $\gamma > 0$.

We will be interested in sampling from $\mu^{(\gamma)}$ via Wilson's algorithm [17]. Consider the network random walk on the graph G , where ordinary edges have weight 1 and dissipative edges have weight γ . This is the Markov chain with state space $\mathbb{Z}^d \cup \{\varpi\}$ and transition probabilities

$$\begin{aligned} p_{x,y} &= \frac{1}{2d + \gamma}, & \text{if } x, y \in \mathbb{Z}^d, |x - y| = 1; \\ p_{x,\varpi} &= \frac{\gamma}{2d + \gamma}, & \text{if } x \in \mathbb{Z}^d. \end{aligned}$$

The Markov chain is stopped at the first time ϖ is hit. The analogous network random walk can also be defined on the graph G_Λ , where again, ordinary edges have weight 1 and dissipative edges have weight γ .

If $\rho = [\rho_0, \rho_1, \dots]$ is a finite path in G_Λ or G , the *loop-erasure* $\text{LE}(\rho)$ of ρ is defined by chronologically erasing loops from the path ρ , as they are created. That is, $\text{LE}(\rho) = [\pi_0, \pi_1, \dots]$, where $\pi_0 := \rho_0$, and for $i \geq 1$ we inductively define

$$\begin{aligned} s_i &:= \max\{n \geq 0 : \rho_n = \pi_{i-1}\} \\ \pi_i &:= \rho_{s_i+1}. \end{aligned}$$

Note that loop-erasure can also be defined for infinite paths ρ that visit any vertex only finitely often. When $(S_n)_{n \geq 0}$ is the network random walk on G_Λ or G , the loop-erasure is called the *Loop-Erased Random Walk* (LERW) [9].

Wilson's algorithm in the case of G_Λ can be stated as follows. Let x_1, \dots, x_N be an enumeration of all vertices in Λ . Let $(S_n^1)_{n \geq 0}, \dots, (S_n^N)_{n \geq 0}$ be independent network random walks started at x_1, \dots, x_N , respectively. We define a growing sequence of trees $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ as follows. Put $\mathcal{F}_0 := \varpi$, and for $i \geq 1$ define inductively

$$\begin{aligned} T^i &:= \inf\{n \geq 0 : S^i(n) \in \mathcal{F}_{i-1}\} \\ \mathcal{F}_i &:= \mathcal{F}_{i-1} \cup \text{LE}(S^i[0, T^i]). \end{aligned} \tag{9}$$

Wilson's Theorem [17] implies that \mathcal{F}_N has distribution $\mu_\Lambda^{(\gamma)}$, irrespective of the chosen enumeration of the vertices. The algorithm also applies to $G_\Lambda^{(0)}$ (this can be obtained by setting $\gamma = 0$, so that dissipative edges are never traversed), and it produces a sample from $\mu_\Lambda^{(0)}$.

Wilson's algorithm in the case of G is similar. We start with an enumeration x_1, x_2, \dots of all vertices of \mathbb{Z}^d , and define the growing sequence of trees $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ as in (9). Put $\mathcal{F} = \cup_{i=0}^\infty \mathcal{F}_i$. By an argument similar to [2, Theorem 5.1] it follows that \mathcal{F} is distributed as $\mu^{(\gamma)}$. Note that due to transience, Wilson's algorithm also makes

sense on $G^{(0)}$ when $d = 3$, and again by [2, Theorem 5.1] it produces a sample from $\mu^{(0)}$.

We now describe the relationship between $\mu^{(\gamma)}$ and $\lim_{\Lambda \uparrow \mathbb{Z}^d} \nu_\Lambda^{(\gamma)}$. Let $\tilde{\Omega} := \{0, 1\}^{E(G)}$. It follows from Wilson's algorithm that when $\gamma > 0$, for $\mu^{(\gamma)}$ -a.e. configuration, for every $x \in \mathbb{Z}^d$ there is a finite path from x to ϖ . Fix an $\omega \in \tilde{\Omega}$ with this property, and let π_x denote the unique self-avoiding path in ω from x to ϖ . We define

$$\begin{aligned} n_x &= \left| \{y \in \mathbb{Z}^d : |y| = 1, d_\omega(x + y, \varpi) < d_\omega(x, \varpi)\} \right| \\ P_x &:= \left\{ y \in \mathbb{Z}^d : |y| = 1, d_\omega(x + y, \varpi) = d_\omega(x, \varpi) - 1 \right\} \\ K_x &:= \{2d - n_x, \dots, 2d - n_x + |P_x| - 1\} \\ y_x &:= \text{the vector such that } (x, x + y_x) \text{ is the first edge of } \pi_x. \end{aligned} \tag{10}$$

Set $\xi_x := \alpha_{P_x, K_x}(y_x)$, $x \in \mathbb{Z}^d$, and define the map $\varphi : \tilde{\Omega} \rightarrow \Omega^{\text{discr},*} = \{0, 1, \dots, 2d - 1, 2d\}^{\mathbb{Z}^d}$ by $\xi =: \varphi(\omega)$. The following lemma is a more explicit version of [7, Lemma 3].

Lemma 2.

- (i) The weak limit $\nu^{(\gamma)} := \lim_{\Lambda \uparrow \mathbb{Z}^d} \nu_\Lambda^{(\gamma)}$ exists for all $\gamma > 0$.
- (ii) The image of $\mu^{(\gamma)}$ under φ is $\nu^{(\gamma)}$ for all $\gamma > 0$.

Proof. (i) Let $x \in \mathbb{Z}^d$, and let x_1, \dots, x_{2d+1} be an enumeration of $\{x + y \in \mathbb{Z}^d : |y| \leq 1\}$. Using Wilson's algorithm, we get that for any $x \in \mathbb{Z}^d$, the joint law of the paths $\{\pi_{x+y, \Lambda}\}_{|y| \leq 1}$ under $\mu_\Lambda^{(\gamma)}$ converges to the joint law of the paths $\{\pi_{x+y}\}_{|y| \leq 1}$ under $\mu^{(\gamma)}$, as $\Lambda \uparrow \mathbb{Z}^d$. It is clear from the definitions of the maps φ_Λ and φ that $\xi_{x, \Lambda}$ and ξ_x only depend on these paths. Therefore, it follows that the law of $\xi_{x, \Lambda}$ under $\nu_\Lambda^{(\gamma)}$ converges to the law of ξ_x under $\mu^{(\gamma)} \circ \varphi^{-1}$.

By essentially the same argument, we obtain that for any finite $B \subset \mathbb{Z}^d$, the joint law of $(\xi_{x, \Lambda})_{x \in B}$ under $\nu_\Lambda^{(\gamma)}$ converges to the joint law of $(\xi_x)_{x \in B}$ under $\mu^{(\gamma)} \circ \varphi^{-1}$. For this we only need to observe that the paths $\{\pi_{x+y, \Lambda}\}_{x \in B, |y| \leq 1}$ determine $(\xi_{x, \Lambda})_{x \in B}$. This proves the weak convergence statement.

(ii) This follows directly from the form of φ_Λ in Section 2.4 and the convergence $\pi_{x, \Lambda} \Rightarrow \pi_x$. \square

Lemma 2 and the uniformity property (3) implies the convergence of the measures $m_\Lambda^{(\gamma)}$. Analogously to the finite Λ case define $\mathcal{X} := \{0, 1, \dots\}^{\mathbb{Z}^d}$, and let $\psi : \mathcal{X} \rightarrow \Omega^{\text{discr},*} = \{0, 1, \dots, 2d - 1, 2d\}^{\mathbb{Z}^d}$ be defined by the same formulae as in (2).

Corollary 1. [7, Lemma 4]

- (i) For any $\gamma \geq 0$ the weak limit $m^{(\gamma)} := \lim_{\Lambda \uparrow \mathbb{Z}^d} m_\Lambda^{(\gamma)}$ exists and the image of $m^{(\gamma)}$ under ψ is $\nu^{(\gamma)}$.
- (ii) The measure $m^{(\gamma)}$ still satisfies the uniformity property (3)

Finally, it was shown in [7, Proposition 4] that as $\gamma \downarrow 0$, $\nu^{(\gamma)} \Rightarrow \nu^{(0)}$ and correspondingly $m^{(\gamma)} \Rightarrow m^{(0)}$. The rest of the paper will give a quantitative version of this statement, in proving Theorem 1.

3 Strategy of the proof

Due to the uniformity property of Corollary 1(ii), it is sufficient to prove Theorem 1 for the measures $\nu^{(\gamma)}$ and $\nu^{(0)}$, in place of $m^{(\gamma)}$ and $m^{(0)}$.

Let $E \subset \Omega^{\text{discr}}$ be a cylinder event depending on heights in $B(k)$, and let w_1, \dots, w_K be a list of all the vertices in $B(k) \cup \partial_{\text{ext}} B(k)$, where $\partial_{\text{ext}} B(k) = \{y \in B(k)^c : \exists z \in B(k) y \sim z\}$. We know that $\mu^{(0)}$ -a.s. there exists $V \in \mathbb{Z}^d$ common to all paths $\pi_{w_i}^{(0)}$, $1 \leq i \leq K$. We select V to be the earliest such vertex with respect to graph distance from $B(k)$.

The key property of the correspondence $\varphi^{(0)}$ given in Section 2.5 is that the height configuration $(\xi_x)_{x \in B(k)} = (\varphi^{(0)}(\omega)_x)_{x \in B(k)}$ only depends on the portion of the paths $\pi_{w_i}^{(0)}$ up to the vertex V .

When γ is small, with high $\mu^{(\gamma)}$ -probability there will be a $V^{(\gamma)} \in \mathbb{Z}^d$ such that all the paths $\pi_{w_i}^{(\gamma)}$ meet at $V^{(\gamma)}$ before they reach ϖ . Our goal is to couple $\mu^{(\gamma)}$ and $\mu^{(0)}$ in such a way that with high probability $V = V^{(\gamma)}$ and $\pi_{w_i}^{(\gamma)} = \pi_{w_i}^{(0)}$ up to the vertex $V^{(\gamma)} = V$.

Consider first $d = 3$. A natural coupling is given by applying Wilson's algorithm with the same random walks for $\mu^{(\gamma)}$ and $\mu^{(0)}$. As mentioned in Section 2.6, the algorithm generalizes to the infinite settings of $\mu^{(0)}$ and $\mu^{(\gamma)}$; see [2, Theorem 5.1]. We start simple random walks at w_1, \dots, w_K . Each random walk receives an independent geometric time of parameter $\lambda = \gamma/(2d + \gamma)$. Ignoring the geometric times, the algorithm realizes a sample from $\mu^{(0)}$. When the random walks are killed at their respective geometric times, the algorithm realizes a sample from $\mu^{(\gamma)}$, with killing corresponding to a jump to ϖ . Let us call this type of coupling of $\mu^{(0)}$ and $\mu^{(\gamma)}$ a *standard coupling*.

It will be convenient to assume the following particular type of enumeration of vertices. Let z_1, \dots, z_N be a list of all vertices in $\partial_{\text{ext}} B(k)$, and let z_{N+1}, \dots, z_{N+M} be a list of all vertices of $B(k)$. In addition we assume that the sequence z_1, \dots, z_N has the property that for each $2 \leq j \leq N$ there exists $1 \leq i(j) < j$ such that $z_{i(j)} \sim z_j$. We denote the spanning tree paths constructed in the coupling by $(\pi_{z_j}^{(0)})_{1 \leq j \leq N+M}$ and $(\pi_{z_j}^{(\gamma)})_{1 \leq j \leq N+M}$, respectively. We will write $\mathbf{B}(x, r)$ for the intersection with \mathbb{Z}^d of the Euclidean ball of radius r centred at x , and $\mathbf{B}(r)$ for $\mathbf{B}(0, r)$. The coupling is successful on the event when the following conditions hold for some $m \geq 2k$:

- (i) for all $2 \leq j \leq N$, $\pi_{z_j}^{(0)}$ intersects $\pi_{z_{i(j)}}^{(0)}$ before leaving $\mathbf{B}(m)$;
- (ii) for all $2 \leq j \leq N$, $\pi_{z_j}^{(\gamma)}$ agrees with $\pi_{z_j}^{(0)}$ up to the first intersection of $\pi_{z_j}^{(0)}$ with

$$\pi_{z_{i(j)}}^{(0)};$$

- (iii) for all $2 \leq j \leq N$, $\pi_{z_{i(j)}}^{(\gamma)}$ agrees with $\pi_{z_{i(j)}}^{(0)}$ up to the last exit of $\pi_{z_{i(j)}}^{(0)}$ from $\mathbf{B}(m)$;
- (iv) for all $N+1 \leq j' \leq N+M$, $\pi_{z_{j'}}^{(\gamma)}$ agrees with $\pi_{z_{j'}}^{(0)}$ up to the first intersection of $\pi_{z_{j'}}^{(0)}$ with $\cup_{1 \leq j \leq N} \pi_{z_j}^{(0)}$.

Note that when (i)–(iii) hold, $\cup_{1 \leq j \leq N} \pi_{z_j}^{(0)}$ separates $B(k)$ from ∞ , and hence condition (iv) makes sense.

We start by reducing the problem of verifying conditions (i)–(iii) to considering just two paths. Let $(S(n))_{n \geq 0}$ and $(S^1(n))_{n \geq 0}$ be independent simple random walks started at the origin and a neighbour of the origin, respectively, and let T and T^1 be independent $\text{Geom}(\lambda)$ and random variables, independent of the walks. Sometimes it will be convenient to allow T^1 to be $\text{Geom}(\lambda_1)$ for some λ_1 . We will write $\mathbf{P}_{\lambda, \lambda_1}(\cdot)$ for the path space measure of the above coupling. Let

$$\begin{aligned} \xi^1 &= \inf\{j \geq 0 : S^1(j) \in \text{LE}(S[0, \infty))\} \\ \xi^{1, \lambda} &= \inf\{j \geq 0 : S^1(j) \in \text{LE}(S[0, T])\}. \end{aligned}$$

Lemma 3. *For any fixed $2 \leq j \leq N$, the joint distribution of*

$$(\pi_{z_{i(j)}}^{(0)}, \pi_{z_{i(j)}}^{(\gamma)}, \pi_{z_j}^{(0)}, \pi_{z_j}^{(\gamma)})$$

is the same as the joint distribution of

$$(\text{LE}(S[0, \infty)), \text{LE}(S[0, T]), \text{LE}(S^1[0, \xi^1]), \text{LE}(S^1[0, \xi^{1, \lambda} \wedge T^1])),$$

up to a shift by $z_{i(j)}$, and a rotation.

Proof. It will be useful to realize the coupling via cycle popping from stacks of arrows, as in [17]. For each $x \in \mathbb{Z}^3$, consider an infinite i.i.d. stack of arrows pointing to random neighbours of x , with the stacks also being independent. In addition, attach to each arrow, independently, a red marker with probability λ . If we ignore the markers, then we have the usual cycle popping in \mathbb{Z}^3 . Now suppose that each arrow that has a marker is replaced by an arrow pointing to ϖ . Then cycle popping realizes the Wilson algorithm for $\mu^{(\gamma)}$.

Consider cycle popping starting at $z_{i(j)}$, that is, follow the arrows, popping each cycle found. This uncovers the path $\pi_{z_{i(j)}}^{(0)}$, and if the red markers are considered, the path $\pi_{z_{i(j)}}^{(\gamma)}$. The paths constructed have the joint distribution of $(\text{LE}(S[0, \infty)), \text{LE}(S[0, T]))$, appropriately shifted. Now continue with cycle popping starting from z_j , which uncovers the paths $\pi_{z_j}^{(0)}$ and $\pi_{z_j}^{(\gamma)}$. Interpreting cycle popping as a random walk, we see that the conditional distribution of the new paths, given the paths already constructed is the same as the conditional distribution of $(\text{LE}(S^1[0, \xi^1]), \text{LE}(S^1[0, \xi^1 \wedge T^1]))$, given $(\text{LE}(S[0, \infty)), \text{LE}(S[0, T^1]))$ (appropriately shifted). \square

Let us write

$$\begin{aligned}\tau_m &= \inf\{j \geq 0 : S(j) \notin \mathbf{B}(m)\} \\ \tau_m^1 &= \inf\{j \geq 0 : S^1(j) \notin \mathbf{B}(m)\}.\end{aligned}$$

Consider the event that the following occur:

- (i') $\tau_m^1 < \xi^1$, that is, S^1 hits $\text{LE}(S[0, \infty))$ before exiting the ball $\mathbf{B}(m)$;
- (ii') $T^1 \geq \tau_m^1$, that is, S^1 is not killed before exiting $\mathbf{B}(m)$;
- (iii') $\text{LE}(S[0, T])$ agrees with $\text{LE}(S[0, \infty))$ up to the last exit of $\text{LE}(S[0, \infty))$ from $\mathbf{B}(m)$.

Then we have the following corollary to Lemma 3.

Corollary 2. *The probability that (i)–(iii) do not all occur is at most $N - 1$ times the probability that (i')–(iii') do not all occur.*

Finally, the probability that (iv) does not occur can be controlled by M times the probability that

$$(iv') \quad T > \tau_{B(k)}.$$

does not occur.

When $d = 2$, $\text{LE}(S[0, \infty))$ is not defined, and we cannot use a standard coupling in the infinite setting. Using a standard coupling in a large finite ball $\mathbf{B}(N)$ is also problematic, due to recurrence, if N is extremely large with respect to λ . Indeed, most of the work in the case $d = 2$ will be to show the existence of a suitable coupling between the paths $\pi_0^{(0)}$ and $\pi_0^{(\gamma)}$. Once this is done, we control the rest of the paths with a Beurling estimate.

Throughout we write C , c , etc. to denote constants whose value may change from line to line.

4 Rate of convergence estimate for $d = 3$

Let $S = \{S(j)\}_{j=0}^\infty$ be a simple random walk started at the origin. Let $\{\hat{S}(j)\}_{j=0}^\infty$ be the loop-erasure of S . We write $\tau_N = \inf\{j \geq 0 : S(j) \notin \mathbf{B}(N)\}$, and $\xi_m = \inf\{j \geq 0 : S(j) \in \mathbf{B}(m)\}$.

We say that $k \geq 0$ is a *cut time* for the random walk S , if $S[0, k] \cap S[k+1, \infty) = \emptyset$. It was shown in [10], that there are constants $c_1, c_2, \zeta_3 > 0$ such that

$$\begin{aligned}c_1 k^{-\zeta_3} &\leq \mathbf{P}[S[0, k] \cap S[k+1, \infty) = \emptyset] \\ &\leq \mathbf{P}[S[0, k] \cap S[k+1, 2k] = \emptyset] \\ &\leq c_2 k^{-\zeta_3}, \quad k \geq 1,\end{aligned}$$

where ζ_3 is called the *intersection exponent* in dimension 3. It was also shown in [10] that with probability bounded away from 0, there are at least $ck^{1-\zeta_3}$ cut times in $[k, 2k]$, and that with R_k denoting the number of cut times in $[0, k]$, $\log R_k / \log k \rightarrow 1 - \zeta_3$ with probability 1. Lawler's proof of the last result also gives an upper bound of $(\log n)^{-C}$ for the probability that there is no cut time in $[\tau_n, \tau_{n(\log n)^c}]$. Essentially the same proof implies the lemma below; see [10, Corollary 4.12].

Lemma 4. *Assume $d = 3$. There exist constants $0 < \beta < 1$, $C < \infty$, such that for $m < n$ we have*

$$\begin{aligned} \mathbf{P}[\text{there is a cut time } k \in [\tau_m, \tau_n] \text{ such that } S[k, \infty) \cap \mathbf{B}(m) = \emptyset] \\ \geq 1 - C(m/n)^\beta. \end{aligned} \quad (11)$$

□

Lemma 5. *Assume $d = 3$. Let $m < n$. There exist constants $C, \alpha < \infty$, such that for $0 < \lambda \leq (m/n)^\beta (\alpha n^2 \log(n/m))^{-1}$ we have*

$$\mathbf{P}[\text{LE}(S[0, T]) \cap \mathbf{B}(m) = \hat{S}[0, \infty) \cap \mathbf{B}(m)] \geq 1 - C(m/n)^\beta, \quad (12)$$

with β as in Lemma 4.

Proof. Consider the event in (11). We claim that on this event, loop-erasure after time τ_n cannot change the intersection of the path with $\mathbf{B}(m)$. Indeed, since the path does not return to $\mathbf{B}(m)$ after time τ_n , the only way such change could occur if a loop touching $\mathbf{B}(m)$ is closed. Any such loop necessarily had to start before the cut time, since after the cut time $\mathbf{B}(m)$ is not visited. However, by definition, the cut time prevents such a loop to be closed.

Since $\mathbf{P}[\tau_n > n^2] \leq c_1 < 1$, there exist constants $c_2 > 0$, $C_2 < \infty$, such that for $x \geq 1$ we have $\mathbf{P}[\tau_n > xn^2] \leq C_2 \exp(-c_1 x)$. In particular, $\mathbf{P}[\tau_n > \alpha n^2 \log(n/m)] \leq C_2(m/n)^{\alpha c_1}$. Choosing $\alpha = \beta/c_1$ makes this bound $C_2(m/n)^\beta$. Now consider

$$\mathbf{P}[T \leq \alpha n^2 \log(n/m)] = 1 - (1 - \lambda)^{\alpha n^2 \log(n/m)} \leq \lambda \alpha n^2 \log(n/m) \leq (m/n)^\beta.$$

When $T > \alpha n^2 \log(n/m)$ and $\tau_n \leq \alpha n^2 \log(n/m)$, we have, $T > \tau_n$, and hence the event in (11) implies the event in (12). The probability that any one of the required events does not occur is at most $C(m/n)^\beta$, as required. □

Let

$$F(\lambda, \lambda_1) = \mathbf{P}[\text{LE}(S[0, T]) \cap S^1[0, T^1] = \emptyset].$$

Write $\tau_n^1 = \inf\{k \geq 0 : S^1(k) \notin \mathbf{B}(n)\}$.

Lemma 6. *Assume $d = 3$. Let $m < n$, $0 < \lambda \leq (m/n)^\beta (\alpha n^2 \log(n/m))^{-1}$. Then there exists constants $C, \delta < \infty$, such that*

$$\mathbf{P}_{\lambda, \lambda}[\text{LE}(S[0, T]) \cap S^1[0, \tau_m^1] = \emptyset] \leq C m^{-1/3} (\delta \log m)^{1/2}. \quad (13)$$

Proof. Let $\lambda_1 = m^{-2}(\delta \log m)^3$. Note that $\lambda < \lambda_1$. Using a large deviation bound for τ_m^1 (see [9, Lemma 1.5.1]) we have

$$\begin{aligned} \mathbf{P}_{\lambda_1}[T^1 > \tau_m^1] &\leq \mathbf{P}_{\lambda_1}[T^1 > m^2/(\delta \log m)^2] + \mathbf{P}[\tau_m^1 \leq m^2/(\delta \log m)^2] \\ &\leq (1 - \lambda_1)^{m^2/(\delta \log m)^2} + C_1 \exp(-\delta \log m) \\ &\leq \frac{C_2}{m^\delta}. \end{aligned}$$

Hence the probability in (13) is at most

$$\begin{aligned} \mathbf{P}_{\lambda, \lambda}[\text{LE}(S[0, T]) \cap S^1[0, \tau_m^1 \wedge T^1] = \emptyset] \\ &\leq \mathbf{P}_{\lambda, \lambda_1}[\text{LE}(S[0, T]) \cap S^1[0, \tau_m^1 \wedge T^1] = \emptyset] \\ &\leq \mathbf{P}_{\lambda, \lambda_1}[\text{LE}(S[0, T]) \cap S^1[0, T^1] = \emptyset] + \mathbf{P}_{\lambda_1}[T^1 > \tau_m^1] \\ &\leq F(\lambda, \lambda_1) + C_2 m^{-\delta} \\ &\leq F(\lambda_1, \lambda_1) + C_2 m^{-\delta}. \end{aligned} \tag{14}$$

It was proved by Lawler [11, Section 12.6], that $F(\lambda_1, \lambda_1) \leq C\lambda_1^{1/6}$. (There the walk S^1 also starts at the origin, however, it is straightforward to deduce the case needed here). Choose $\delta > 1/3$. Then the right hand side of (14) is at most $Cm^{-1/3}(\delta \log m)^{1/2}$. \square

Proof of Theorem 1; $d = 3$. We choose $n = m^{1+(3\beta)^{-1}}$ in Lemmas 5 and 6, which makes the upper bounds in those lemmas $m^{-1/3+o(1)} = \lambda^{\beta(2+7\beta)^{-1}+o(1)}$.

Let z_1, \dots, z_N be an enumeration of the vertices in $\partial_{\text{ext}} B(k)$, and z_{N+1}, \dots, z_{N+M} an enumeration of the vertices of $B(k)$. We assume that for each $2 \leq j \leq N$ there exists $1 \leq i(j) < j$ such that $z_{i(j)} \sim z_j$. Note that when the events in the statements of Lemmas 5 and 6 occur, then (i')–(iii') occur. Hence the probability that there exists $2 \leq j \leq N$ for which (i)–(iii) do not occur is at most $Ck^2\lambda^{\beta(2+7\beta)^{-1}+o(1)}$. Note that the union of the paths $\cup_{i=1}^N \pi_{z_i}^{(0)}$ disconnects $B(k)$ from $\partial \mathbf{B}(m)$. Hence the random walks S^i , $i = N+1, \dots, N+M$ necessarily hit the earlier paths. It follows that the probability that (iv) does not occur for some $N+1 \leq i \leq N+M$ is at most M times the probability $\mathbf{P}[T < \tau_m]$. This can be bounded as $Ck^3\lambda(\log k)k^2$. Therefore, Theorem 1 follows in the case $d = 3$. \square

5 Rate of convergence estimate for $d = 2$

The proof in the case $d = 2$ follows a somewhat different outline, for two reasons. First, the loop-erasure of $S[0, \infty)$ cannot be defined due to recurrence, so it has to be replaced by the infinite loop-erased random walk [9, Section 7.4], leading to a different coupling. Second, there are no global cut-times, so we will work with a finite volume analogue.

Here is the outline of the proof. We first couple a suitable initial segment of $\text{LE}(S[0, T])$ to an initial segment of the infinite LERW, and show that for suitable

m , with high probability, $\text{LE}(S[0, T]) \cap \mathbf{B}(m)$ is determined by this initial segment alone. This will give the required coupling between the paths starting at 0 for the measures $\mu^{(\gamma)}$ and $\mu^{(0)}$. Then we use Wilson's method to generate the paths starting in $B(k) \cup \partial_{\text{ext}} B(k)$, using the same random walks for $\mu^{(\gamma)}$ and $\mu^{(0)}$. We show that with high probability the new paths all stay inside $\mathbf{B}(m)$ and are not killed before their respective hitting times.

Let \hat{S}^λ , respectively \hat{S}^N , denote the loop-erasures of $S[0, T]$, respectively $S[0, \tau_N]$. Let Γ_r denote the set of r -step self-avoiding paths starting at 0. We define the measures $\hat{P}^\lambda = \hat{P}_r^\lambda$ and $\hat{P}^N = \hat{P}_r^N$ on Γ_r by the formulas

$$\hat{P}^\lambda(\gamma) = \mathbf{P} \left[\hat{S}^\lambda[0, r] = \gamma \right] \quad \text{and} \quad \hat{P}^N(\gamma) = \mathbf{P} \left[\hat{S}^N[0, r] = \gamma \right].$$

Note that \hat{P}_r^λ is not a probability measure, since $T < r$ has positive probability. It was shown in [9, Section 7.4] that for every $r \geq 1$ the limit $\lim_{N \rightarrow \infty} \hat{P}_r^N =: \hat{P}_r$ exists, and hence \hat{S}^N converges weakly to a limit $\hat{S}[0, \infty)$, called the infinite LERW.

Let $\xi_A = \inf\{j \geq 1 : S(j) \in A\}$. Below we write $\mathbf{P}^y[\cdot]$ for probability under which S starts at y , and the clock of T starts at 0.

The following lemma can be proved similarly to [9, Proposition 7.3.1].

Lemma 7. *If $\gamma_r = [\gamma(0), \dots, \gamma(r)] \in \Gamma_r$, $r \geq 1$, and $\gamma_{r-1} = [\gamma(0), \dots, \gamma(r-1)]$, then*

$$\hat{P}^\lambda(\gamma_r) = \hat{P}^\lambda(\gamma_{r-1}) \mathbf{P}^{\gamma(r-1)}[S(1) = \gamma(r), T \geq 1 \mid \xi_A > T],$$

where $A = \{\gamma(0), \dots, \gamma(r-1)\}$. Also,

$$\mathbf{P}[\hat{S}^\lambda = \gamma_{r-1}] = \hat{P}^\lambda(\gamma_{r-1}) \mathbf{P}^{\gamma(r-1)}[T = 0 \mid \xi_A > T].$$

The lemma implies that given the first $r-1$ steps of the loop-erased walk, the r -th step of the walk will be to $y \sim \gamma(r-1)$, $y \notin A$, with probability proportional to

$$\frac{1-\lambda}{4} \mathbf{P}^y[\xi_A > T], \tag{15}$$

and that with probability proportional to λ the walk has no r -th step.

Our first goal is to show that when λ is small, a suitable initial segment of \hat{S}^λ can be coupled to the corresponding initial segment of \hat{S} with high probability. This will be achieved by adapting the proof of [9, Proposition 7.4.2]. We start by identifying a set of “good” paths, where the probabilities in (15) will behave sufficiently regularly.

Let \mathcal{T}^N denote the wired uniform spanning tree in $\mathbf{B}(N)$ [2]. Due to Wilson's algorithm, the distribution of \hat{S}^N is the same as the distribution of the path γ^N connecting 0 to the wired vertex in \mathcal{T}^N . For $1 \leq R < N$, and $x \in \mathbf{B}(R)$, let $\gamma^{x,N}$ denote the unique self-avoiding path in \mathcal{T}^N connecting x to 0. Let $\beta > 0$ be a parameter, whose value will be fixed later. We will measure the “badness” of paths through the random variable

$$X_R^{(N)} := \#\{x \in \mathbf{B}(R) : \gamma^{x,N} \subset \mathbf{B}(R), x \in \gamma^N, \exists y \sim x \mathbf{P}^y[\xi_{\gamma^x} > \tau_{2R}] \leq R^{-\beta}\}.$$

Lemma 8. Assume $d = 2$. Suppose we have $\gamma = [\gamma(0), \dots, \gamma(\ell)]$, $\gamma \subset \mathbf{B}(R)$, and $y_1, y_2 \sim \gamma(\ell)$ such that $\mathbf{P}^{y_i}[\xi_\gamma > \tau_{2R}] > 0$, $i = 1, 2$. Then

$$\mathbf{P}^{y_1}[\xi_\gamma > \tau_{2R}] \geq \frac{1}{256} \mathbf{P}^{y_2}[\xi_\gamma > \tau_{2R}].$$

Proof. It is sufficient to show that there exists a path of at most 4 steps from y_1 to y_2 that avoids γ . It follows from the planarity of the configuration that such a path exists. \square

Lemma 9. There exists $C < \infty$, such that for all $N > 2R$ we have $\mathbf{P}[X_R^{(N)} \geq 1] \leq C(\log R)R^{2-\beta}$.

Proof. We use Wilson's algorithm rooted at 0. We write ϖ for the wired vertex of the graph $G_{\mathbf{B}(N)}^{(0)}$ obtained from $\mathbf{B}(N)$. First generate the path $\gamma^{x,N}$ by running a LERW from x to 0. Next run a LERW from ϖ . Then

$$\begin{aligned} \mathbb{E}[X_R^{(N)}] &= \sum_{x \in \mathbf{B}(R)} \mathbb{E} \left[\mathbf{P}^\varpi[\text{first hit } \gamma^{x,N} \text{ at } x] I[\gamma^{x,N} \subset \mathbf{B}(R)] \right. \\ &\quad \left. I[\exists y \sim x : 0 < \mathbf{P}^y[\xi_{\gamma^{x,N}} > \tau_{2R}] \leq R^{-\beta}] \right]. \end{aligned} \quad (16)$$

Let $G^N(u, v)$ denote the Green function of random walk on the wired graph $G_{\mathbf{B}(N)}^{(0)}$, killed upon hitting 0, and let $G^{\gamma^{x,N}, N}(u, v)$ denote the Green function of random walk on the same graph, killed upon hitting $\gamma^{x,N}$. Let $D_N = \text{degree of } \varpi$, and note that $D_N \geq cN$. Using reversibility of the random walk, we have

$$\begin{aligned} \mathbf{P}^\varpi[\text{first hit } \gamma^{x,N} \text{ at } x] &= \frac{4}{D_N} G^{\gamma^{x,N}, N}(\varpi, \varpi) \mathbf{P}^x[\text{no return to } \gamma^{x,N} \text{ before } \tau_N] \\ &= \frac{4}{D_N} G^{\gamma^{x,N}, N}(\varpi, \varpi) \frac{1}{4} \sum_{y \sim x} \mathbf{P}^y[\xi_{\gamma^{x,N}} > \tau_N]. \end{aligned} \quad (17)$$

We have $G^{\gamma^{x,N}, N}(\varpi, \varpi) \leq G^N(\varpi, \varpi)$. For u in the interior boundary of $\mathbf{B}(N)$, the probability that a random walk started at u hits 0 before exiting $\mathbf{B}(N)$ is bounded below by $c/(N \log N)$, with c independent of u and N . This follows from the facts that there is probability at least c/N for the walk to reach $\mathbf{B}(N/2)$ before exiting $\mathbf{B}(N)$, and there is probability $c/\log N$ for it to reach 0 from $\partial\mathbf{B}(N/2)$ before exiting $\mathbf{B}(N)$ (see [9, Exercise 1.6.8]). This implies that $G^N(\varpi, \varpi) \leq CN \log N$. Each term in the sum over y is either 0, or can be bounded by

$$\mathbf{P}^y[\xi_{\gamma^{x,N}} > \tau_{2R}] \sup_{z \in \partial\mathbf{B}(2R)} \mathbf{P}^z[\xi_0 > \tau_N] \leq CR^{-\beta} \frac{\log R}{\log N}.$$

due to Lemma 8. We obtain that the right hand side of (17) is less than or equal to

$$\frac{C}{D_N} G^N(\varpi, \varpi) \frac{\log R}{\log N} R^{-\beta} \leq C(\log R)R^{-\beta}.$$

Inserting this into (16), we obtain the statement of the lemma. \square

We will require that $\beta > 2$. We introduce the length scale n of the form $n = \lambda^{-\rho}$, where the exponent ρ will be chosen at the end of the proof. Its role will be to allow us to replace the killing time T by τ_n in such a way that with high probability T occurs later than τ_n , but not much later (up to a power). We will choose R of the form $R = \lambda^{-\rho'}$, with $\rho' < \rho < 1/2$, so that $R \ll n \ll 1/\sqrt{\lambda}$, and ρ' will also be chosen at the end of the proof. In the rest of this section, c, c', C, C' may depend on the exponents ρ, ρ' , etc., but do not depend on λ .

Proposition 1. *Suppose that $1 - 2\rho > (1 + \beta)\rho'$ and $2\rho' < \rho$. There exists a subset $\mathcal{P} \subset \Gamma_R$ such that $\hat{P}(\mathcal{P}) \geq 1 - C(\log R)R^{2-\beta}$ and for all $\gamma \in \mathcal{P}$ we have*

$$\hat{P}^\lambda(\gamma) = \hat{P}(\gamma) \left(1 + O\left(\lambda n^2 R^{1+\beta} / \log[\lambda^{-1}]\right) + O\left(\frac{R^2}{n} \log \frac{n}{R}\right) \right).$$

Proof. The proof goes by adapting the argument of [9, Proposition 7.4.2]. The main difference from that proof is that “trapping” can occur, and convergence does not hold uniformly over all paths.

Let \mathcal{P} be the set of paths $\gamma = [\gamma(0), \dots, \gamma(R)]$ such that for all $1 \leq \ell \leq R$ we have $\mathbf{P}^{\gamma(\ell)}[\xi_{\gamma_\ell} > \tau_{2R}] \geq R^{-\beta}$, with $\gamma_\ell := [\gamma(0), \dots, \gamma(\ell)]$ and for all $y \sim \gamma(\ell - 1)$, $y \notin \gamma_\ell$ we have either $\mathbf{P}^y[\xi_{\gamma_\ell} > \tau_{2R}] = 0$ or $\mathbf{P}^y[\xi_{\gamma_\ell} > \tau_{2R}] \geq R^{-\beta}$. Then Lemma 9 and Lemma 8 show that

$$\hat{P}(\mathcal{P}) = \lim_{N \rightarrow \infty} \hat{P}^N(\mathcal{P}) \geq 1 - \limsup_{N \rightarrow \infty} \mathbf{P}[X_R^{(N)} \geq 1] \geq 1 - C(\log R)R^{2-\beta}.$$

For $1 \leq j \leq R$, Lemma 7 implies

$$\hat{P}^\lambda(\gamma_j) = \hat{P}^\lambda(\gamma_{j-1}) \frac{\frac{1-\lambda}{4} \mathbf{P}^{\gamma(j)}[\xi_A > T]}{\lambda + \frac{1-\lambda}{4} \sum_{y \notin A, y \sim \gamma(j-1)} \mathbf{P}^y[\xi_A > T]}, \quad (18)$$

where $A = A_j = \{\gamma(0), \dots, \gamma(j-1)\}$. The corresponding formula for \hat{P}^n is [9, Eqn. (7.3)]:

$$\hat{P}^n(\gamma_j) = \hat{P}^n(\gamma_{j-1}) \frac{\mathbf{P}^{\gamma(j)}[\xi_A > \tau_n]}{\sum_{y \notin A, y \sim \gamma(j-1)} \mathbf{P}^y[\xi_A > \tau_n]}.$$

Assuming $\gamma \in \mathcal{P}$, and $1 \leq j \leq R$, we start by relating $\mathbf{P}^y[\xi_A > T]$ to $\mathbf{P}^y[\xi_A > \tau_n]$. We can write

$$\begin{aligned} \mathbf{P}^y[\xi_A > T] &= \mathbf{P}^y[\xi_A > T, T < \tau_n] \\ &\quad + \mathbf{P}^y[\xi_A > \tau_n, T \geq \tau_n] \mathbf{P}^y[\xi_A > T | \xi_A > \tau_n, T \geq \tau_n]. \end{aligned} \quad (19)$$

The first term on the right hand side of (19) will be an error term, that we estimate as follows. Let $\alpha > 0$ be a large parameter, that we will choose in the course of the proof.

$$\begin{aligned} \mathbf{P}^y[\xi_A > T, T < \tau_n] &\leq \mathbf{P}^y[\tau_n > \alpha(\log n)n^2] + \mathbf{P}^y[T < \alpha(\log n)n^2] \\ &\leq \exp(-c\alpha \log n) + \lambda\alpha(\log n)n^2 \\ &\leq C\lambda(\log n)n^2. \end{aligned} \quad (20)$$

The last step is justified, if we take $\alpha > (\frac{1}{\rho} - 2)/c$.

Let us consider the second term on the right hand side of (19). Using the strong Markov property, and the memoryless property of T , we can write

$$\begin{aligned} & \mathbf{P}^y[\xi_A > T | \xi_A > \tau_n, T \geq \tau_n] \\ &= \sum_{z \in \partial \mathbf{B}(n)} \mathbf{P}^y[S(\tau_n) = z | \xi_A > \tau_n, T \geq \tau_n] \mathbf{P}^z[\xi_A > T]. \end{aligned} \quad (21)$$

Let $H_n(z) := \mathbf{P}^0[S(\tau_n) = z]$. The key step of the proof is to show that the first factor of the summand on the right hand side of (21) is essentially independent of y , and equals $H_n(z)(1 + O(\frac{R}{n} \log \frac{n}{R}) + O(\lambda(\log n)n^2))$. Using the strong Markov property, and the memoryless property of T , we can write

$$\begin{aligned} & \mathbf{P}^y[\xi_A > \tau_n, T \geq \tau_n, S(\tau_n) = z] \\ &= \sum_{w \in \partial \mathbf{B}(2R)} \mathbf{P}^y[\xi_A > \tau_{2R}, T \geq \tau_{2R}, S(\tau_{2R}) = w] \\ & \quad \times \mathbf{P}^w[\xi_A > \tau_n, T \geq \tau_n, S(\tau_n) = z]. \end{aligned} \quad (22)$$

By [9, Eqn. (2.10)], we have

$$\mathbf{P}^w[\xi_A > \tau_n, S(\tau_n) = z] = \mathbf{P}^w[\xi_A > \tau_n] H_n(z) \left(1 + O\left(\frac{R}{n} \log \frac{n}{R}\right) \right). \quad (23)$$

Hence we want to estimate the effect of omitting the event $T \geq \tau_n$ from the second probability on the right hand side of (22). By a standard estimate [9, Exercise 1.6.8], we have $\mathbf{P}^w[\xi_A > \tau_n] \geq \mathbf{P}^w[\xi_{\mathbf{B}(R)} > \tau_n] \geq c[\log \frac{n}{R}]^{-1}$. We also have $H_n(z) \geq cn^{-1}$, by [9, Lemma 1.7.4]. This implies that

$$\mathbf{P}^w[\xi_A > \tau_n, S(\tau_n) = z] \geq c \left(n \log \frac{n}{R} \right)^{-1}. \quad (24)$$

We are now ready to estimate

$$\begin{aligned} & \mathbf{P}^w[\xi_A > \tau_n, T < \tau_n, S(\tau_n) = z] \\ & \leq \mathbf{P}^w[\tau_n > \alpha(\log n)n^2] + \mathbf{P}^w[\xi_A > \tau_n, T < \alpha(\log n)n^2, S(\tau_n) = z] \\ & \leq \exp(-c\alpha \log n) + \lambda\alpha(\log n)n^2 \mathbf{P}^w[\xi_A > \tau_n, S(\tau_n) = z] \\ & \leq \mathbf{P}^w[\xi_A > \tau_n, S(\tau_n) = z] O(\lambda(\log n)n^2). \end{aligned} \quad (25)$$

Here we have used the lower bound (24) and we require that α satisfy $\alpha > (\frac{1}{\rho} - 1)/c$. Putting the estimates (23) and (25) together we have

$$\begin{aligned} \mathbf{P}^w[\xi_A > \tau_n, T \geq \tau_n, S(\tau_n) = z] &= \mathbf{P}^w[\xi_A > \tau_n] H_n(z) \\ & \quad \times \left(1 + O\left(\frac{R}{n} \log \frac{n}{R}\right) + O(\lambda(\log n)n^2) \right). \end{aligned}$$

Putting this back into (22), we get the key estimate:

$$\mathbf{P}^y[S(\tau_n) = z | \xi_A > \tau_n, T \geq \tau_n] = H_n(z) \left(1 + O\left(\frac{R}{n} \log \frac{n}{R}\right) + O(\lambda(\log n)n^2) \right). \quad (26)$$

Inserting (26) into (21) and summing over z , we get

$$\mathbf{P}^y[\xi_A > T | \xi_A > \tau_n, T \geq \tau_n] = D(\lambda, n, A) + O\left(\frac{R}{n} \log \frac{n}{R}\right) + O(\lambda(\log n)n^2), \quad (27)$$

where $D(\lambda, n, A) = \sum_{z \in \partial \mathbf{B}(n)} H_n(z) \mathbf{P}^z[\xi_A > T]$.

Now we are ready to start analyzing the expression (18). Consider $y \sim \gamma(j-1)$, $y \notin A$, such that $\mathbf{P}^y[\xi_A > \tau_{2R}] > 0$. The main contribution to $\mathbf{P}^y[\xi_A > T]$ will be $\mathbf{P}^y[\xi_A > T, \xi_A > \tau_n]$. An estimate very similar to (20) yields

$$\mathbf{P}^y[\xi_A > T, \xi_A < \tau_n] \leq C\lambda(\log n)n^2. \quad (28)$$

A lower bound for the main term is as follows.

$$\begin{aligned} \mathbf{P}^y[\xi_A > T, \xi_A > \tau_n] \\ \geq \mathbf{P}^y[\xi_A > \tau_{2R}] \mathbf{P}^y[\xi_A > \tau_{1/\sqrt{\lambda}} | \xi_A > \tau_{2R}] \mathbf{P}^y[\xi_A > T | \xi_A > \tau_{1/\sqrt{\lambda}}]. \end{aligned} \quad (29)$$

The first factor on the right hand side of (29) is at least $cR^{-\beta}$, due to $\gamma \in \mathcal{P}$. The second factor is at least the minimum over z of the probability that random walk started at $z \in \partial \mathbf{B}(2R)$ will exit $\mathbf{B}(1/\sqrt{\lambda})$ before hitting $\mathbf{B}(R)$. This is at least $c \log(2R/R) / \log[(\sqrt{\lambda}R)^{-1}] > c' / \log[\lambda^{-1}]$. We show that the third factor is bounded away from 0. Due to the invariance principle, with probability bounded away from 0, a random walk started on the boundary of $\mathbf{B}(1/\sqrt{\lambda})$ will take at least $1/\lambda$ steps before hitting $\mathbf{B}(1/(2\sqrt{\lambda}))$, and hence before hitting A . On this event, the conditional probability of $\xi_A > T$ is at least $P[T \leq \lambda^{-1}] \approx 1 - e^{-1}$. Putting the estimates together, we get

$$\mathbf{P}^y[\xi_A > T, \xi_A > \tau_n] \geq cR^{-\beta} / \log[\lambda^{-1}]. \quad (30)$$

The choice $\lambda = n^{-\rho}$ and the condition on ρ and ρ' ensure that $C\lambda(\log n)n^2$ is of smaller order than the right hand side of (30). Putting (28) and (30) together, we obtain

$$\mathbf{P}^y[\xi_A > T] \geq cR^{-\beta} / \log[\lambda^{-1}]. \quad (31)$$

Arguments similar to what led to (31), also yield the simple lower bound

$$D(\lambda, n, A) \geq c. \quad (32)$$

We return to (19). The estimates (27), (28), (31) and (32) imply

$$\begin{aligned} \mathbf{P}^y[\xi_A > T] &= O(\lambda(\log n)n^2) + (\mathbf{P}^y[\xi_A > \tau_n] + O(\lambda(\log n)n^2)) \\ &\quad \times \left(D(\lambda, n, A_j) + O\left(\frac{R}{n} \log \frac{n}{R}\right) + O(\lambda(\log n)n^2) \right) \\ &= \mathbf{P}^y[\xi_A > \tau_n] \left(D(\lambda, n, A_j) + O\left(\frac{R}{n} \log \frac{n}{R}\right) + O\left(\lambda n^2 R^\beta / \log[\lambda^{-1}]\right) \right). \end{aligned} \quad (33)$$

Consider now the case when $y \sim \gamma(j-1)$, $y \notin A_j$, such that $\mathbf{P}^y[\xi_A > \tau_{2R}] = 0$. In this case we have

$$\mathbf{P}^y[\xi_A > T] \leq \mathbf{P}^y[T < \tau_{2R}] \leq C\lambda\alpha(\log n)n^2. \quad (34)$$

Putting (33) and (34) into (18), we get

$$\hat{P}^\lambda(\gamma_j) = \hat{P}^\lambda(\gamma_{j-1}) \frac{\hat{P}^n(\gamma_j)}{\hat{P}^n(\gamma_{j-1})} \left(1 + O(\lambda n^2 R^\beta / \log[\lambda^{-1}]) + O\left(\frac{R}{n} \log \frac{n}{R}\right) \right). \quad (35)$$

Iterating for $j = 1, \dots, R$ we get

$$\hat{P}^\lambda(\gamma) = \hat{P}^n(\gamma) \left(1 + O(\lambda n^2 R^{1+\beta} / \log[\lambda^{-1}]) + O\left(\frac{R^2}{n} \log \frac{n}{R}\right) \right).$$

The proposition follows, since due to [9, Proposition 7.4.2], we have

$$\hat{P}^n(\gamma) = \hat{P}(\gamma) \left(1 + O\left(\frac{R^2}{n} \log \frac{n}{R}\right) \right).$$

□

Our next step is to get an estimate on the probability that $\hat{S} \cap \mathbf{B}(m)$ differs from $\text{LE}(S[0, T]) \cap \mathbf{B}(m)$ for suitable m . We will select m of the form $m = \lambda^{-\rho''}$, and require $\rho'' < \rho'$, so that $m \ll R$. We will need the discrete Beurling estimate, stated below. This was first proved by Kesten [8]; see [12] for a version more similar to what will be used here.

Theorem 2 (Beurling estimate, [12]). *Suppose $m < N$, and $A \subset \mathbb{Z}^2$ contains a path from $\mathbf{B}(m)$ to $\partial\mathbf{B}(N)$. There exists a constant $C < \infty$ such that for any $z \in \partial\mathbf{B}(m)$ we have*

$$\mathbf{P}^z[\tau_N < \xi_A] \leq C(m/N)^{1/2}.$$

The next proposition estimates the probability that the loop-erasure inside a ball $\mathbf{B}(m)$ is affected after the loop-erased path has reached distance $R' > m$. Later on we are going to take $R' = (1/2)\sqrt{R}$ with R as in Proposition 1. We define

$$\begin{aligned} \hat{\tau}_{R'}^\lambda &= \inf\{j \geq 0 : \hat{S}^\lambda(j) \in \partial\mathbf{B}(R')\} \\ \hat{\tau}_{R'} &= \inf\{j \geq 0 : \hat{S}(j) \in \partial\mathbf{B}(R')\}. \end{aligned}$$

Proposition 2. *Assume $d = 2$. Let $16m < R'$. Suppose that $\lambda \leq (R')^{-4}$. There exists a constant $C < \infty$ such that*

$$\mathbf{P}[\hat{S}^\lambda(j) \notin \mathbf{B}(m) \text{ for } j \geq \hat{\tau}_{R'}^\lambda] \geq 1 - C(m/R') \log(R'/m). \quad (36)$$

Likewise, we have

$$\mathbf{P}[\hat{S}(j) \notin \mathbf{B}(m) \text{ for } j \geq \hat{\tau}_{R'}] \geq 1 - C(m/k') \log(k'/m). \quad (37)$$

Proof. We estimate the probability that after the loop-erasure of $S[0, T]$ has reached $\partial\mathbf{B}(R')$, the walk S revisits $\mathbf{B}(m)$, before time T . Condition on the set $A = \hat{S}^\lambda[0, \hat{\tau}_{R'}^\lambda - 1]$, and let $x = \hat{S}^\lambda(\hat{\tau}_{R'}^\lambda)$. Let

$$\rho = \max\{j < T : S(j) = S(\hat{\tau}_{R'}^\lambda - 1)\}.$$

The law of $S[\rho+1, T]$ is that of a random walk started at x and conditioned on $\xi_A > T$. We show that

$$\mathbf{P}^x[\xi_{\mathbf{B}(m)} \leq T | \xi_A > T] \leq C_1(m/R') \log(R'/m), \quad (38)$$

which implies the claim of the proposition. The walk first has to exit $\mathbf{B}(x, R'/4)$ without hitting A . Then it has to cross $\mathbf{B}(R'/4) \setminus \mathbf{B}(m)$ without hitting A , in order to visit $\mathbf{B}(m)$. Both of these have to occur before time T . Following the visit to $\mathbf{B}(m)$, the walk has to avoid A until time T . Since $1/\sqrt{\lambda}$ is of larger order than R' , this means that the walk will essentially have to cross $\mathbf{B}(R'/4) \setminus \mathbf{B}(m)$ again without hitting A , and stay away from A after that. Since A contains a path from $\mathbf{B}(m)$ to $\partial\mathbf{B}(R'/4)$, the Beurling estimate Theorem 2 can be used to bound the probability that A is not hit during the crossings. This will yield a bound $C_2(m/R')^{1/2}(m/R')^{1/2} \log(R'/m)$, where the log-factor arises due to a technicality. The probability of not hitting A during the final stretch will yield a factor $C/\log(1/\sqrt{\lambda}R')$, that will be used to cancel the effect of the conditioning.

We first get a lower bound for the probability of the conditioning in (38). Let c_1 be a constant such that the set $\{y \in \partial\mathbf{B}(x, R'/4) : |y| > (1 + c_1)R'\}$ contains at least a fraction $c_2 > 0$ of $\partial\mathbf{B}(x, R'/4)$. Let us write $\tau_{x, R'}$ for $\tau_{\mathbf{B}(x, R'/4)}$. We have

$$\begin{aligned} \mathbf{P}^x[\xi_A > \tau_{x, R'}, T \geq \tau_{x, R'}, |S(\tau_{x, R'})| > (1 + c_1)R'] \\ \geq \mathbf{P}^x[\xi_A > \tau_{x, R'}, |S(\tau_{x, R'})| > (1 + c_1)R'] \\ - \mathbf{P}^x[\tau_{x, R'} > \alpha(\log R')(R')^2] \\ - \mathbf{P}^x[T < \alpha(\log R')(R')^2]. \end{aligned} \quad (39)$$

We claim that the subtracted terms in (39) are of lower order than the first term. The first term is at least the probability that S exists $\mathbf{B}(x, R'/4)$ without returning to $\mathbf{B}(R')$, and reaches a distance of order R' from $\mathbf{B}(R')$. This probability is at least c/R' [9, Exercise 1.6.8]. The second term is at most $\exp(-c\alpha \log R')$, which for α large enough is of smaller order than the first term. The third term is $O(\lambda(\log R')(R')^2) = o(1/R')$, by the condition on λ . It is intuitive that given the event $\xi_A > \tau_{x, R'}$, we have $|S(\tau_{x, R'})| > (1 + c_1)R'$ with conditional probability bounded away from 0. A proof of this can be found in [14, Proposition 3.5]. Hence we have

$$\mathbf{P}^x[\xi_A > \tau_{x, R'}, T \geq \tau_{x, R'}, |S(\tau_{x, R'})| > (1 + c_1)R'] \geq c\mathbf{P}^x[\xi_A > \tau_{x, R'}]. \quad (40)$$

The walk having reached distance $(1 + c_1)R'$, the probability that the walk will avoid A until time T is at least the probability that (i) it exits $\mathbf{B}(2/\sqrt{\lambda})$ without hitting A ;

and (ii) it takes at least $1/\lambda$ steps before entering $\mathbf{B}(1/\sqrt{\lambda})$; and (iii) $T \leq 1/\lambda$. The probability of (i) is at least $c \log(2(\sqrt{\lambda}R')^{-1})$. The probability of (ii) is at least c , due to the invariance principle. The probability of (iii) is bounded away from 0. This gives us

$$\mathbf{P}^x[\xi_A > T] \geq c\mathbf{P}^x[\xi_A > \tau_{x,R'}] \log(2(\sqrt{\lambda}R')^{-1}). \quad (41)$$

We now come to deriving an upper bound for $\mathbf{P}^x[\xi_{\mathbf{B}(m)} \leq T, \xi_A > T]$. First consider the walk up to its first exit from $\mathbf{B}(x, R'/4)$. Then we see that the event $\{\xi_A > \tau_{x,R'}, T \geq \tau_{x,R'}\}$ has to occur. We neglect the requirement that $T \geq \tau_{x,R'}$, and use $\mathbf{P}^x[\xi_A > \tau_{x,R'}]$ as an upper bound. We now consider the walk starting on $\partial\mathbf{B}(x, R'/4)$. We need to make precise the estimates on the probability of crossing $\mathbf{B}(R'/4) \setminus \mathbf{B}(m)$. For the first crossing, we can again neglect the event $T \geq \xi_{\mathbf{B}(m)}$. We introduce the notation

$$\begin{aligned} \xi_1 &= \inf\{j \geq 0 : S(j) \in \partial\mathbf{B}(R'/4)\} \\ \xi_2 &= \inf\{j \geq \xi_1 : S(j) \in \mathbf{B}(m)\} \\ \bar{\xi}_1 &= \sup\{\xi_1 \leq j \leq \xi_2 : S(j) \in \partial\mathbf{B}(R'/4)\} \\ Y &= S(\bar{\xi}_1) \\ Z &= S(\xi_2) \\ \bar{\xi}_1' &= \sup\{\xi_1 \leq j \leq \xi_2 : S(j) \in \partial\mathbf{B}(R'/8)\} \\ \bar{\xi}_2' &= \sup\{\xi_1 \leq j \leq \xi_2 : S(j) \in \partial\mathbf{B}(2m)\} \\ Z' &= S(\bar{\xi}_2'). \end{aligned}$$

Fix $y \in \partial\mathbf{B}(R'/4)$, $z \in \mathbf{B}(m)$, and $z' \in \partial\mathbf{B}(2m)$. Let \tilde{S} denote a random walk started at z . We will use tildes for stopping times corresponding to \tilde{S} . Conditioned on the event $\{\xi_2 < \infty, Y = y, Z = z, Z' = z'\}$, the time reversal of $S[\bar{\xi}_1', \bar{\xi}_2']$ has the same law as $\tilde{S}[\tilde{\tau}_{2m}, \tilde{\tau}_{R'/8}]$ conditioned on the event

$$\tilde{E}_y = \{\tilde{S}(\tilde{\tau}_{2m}) = z', \tilde{\tau}_{R'/4} < \tilde{\xi}_{\mathbf{B}(m)}, \tilde{S}(\tilde{\tau}_{R'/4}) = y\}.$$

We have

$$\mathbf{P}^{z'}[\tilde{\tau}_{R'/4} < \tilde{\xi}_{\mathbf{B}(m)}] \geq \frac{C_1}{\log(R'/m)}.$$

Due to the Harnack principle [9, Theorem 1.7.6], conditioning on $\tilde{S}(\tilde{\tau}_{R'/4}) = y$ affects the probability of $\{\tilde{S}[\tilde{\tau}_{2m}, \tilde{\tau}_{R'/8}] \cap A = \emptyset\}$ by a factor that is bounded away from 0 and ∞ . Hence it follows that

$$\begin{aligned} &\mathbf{P}[\xi_2 < \infty, S[\xi_1, \xi_2] \cap A = \emptyset] \\ &\leq \sup_{z', y} \mathbf{P}^{z'}[\tilde{S}[0, \tilde{\tau}_{R'/8}] \cap A = \emptyset | \tilde{E}_y] \\ &\leq C_1 \log(R'/m) \sup_{z'} \mathbf{P}^{z'}[\tilde{S}[0, \tilde{\tau}_{R'/8}] \cap A = \emptyset | \tilde{S}(\tilde{\tau}_{R'/4}) = y] \\ &\leq C_2 \log(R'/m) \sup_{z'} \mathbf{P}^{z'}[\tilde{S}[0, \tilde{\tau}_{R'/8}] \cap A = \emptyset] \\ &\leq C_3 (m/R')^{1/2} \log(R'/m). \end{aligned} \quad (42)$$

In the last step, we use Theorem 2.

For the other crossing of $\mathbf{B}(R'/4) \setminus \mathbf{B}(m)$ we apply Theorem 2 directly. Due to the memoryless property of T , we may assume that the clock of T is starting at time ξ_2 . Let

$$\tau_1 = \inf\{j \geq \xi_2 : S(j) \notin \mathbf{B}(R'/4)\}.$$

Then

$$\begin{aligned} \mathbf{P}[S[\xi_2, \tau_1] \cap A = \emptyset | \xi_2 < \infty] &\leq \sup_{z \in \partial \mathbf{B}(m)} \mathbf{P}^z[S[0, \tau_{R'/4}] \cap A = \emptyset] \\ &\leq C_4(m/R')^{1/2}. \end{aligned} \tag{43}$$

The probability of $T < \tau_1$ can be estimated similarly to (39), and is of smaller order than the right hand side of (43). Therefore we also have

$$\mathbf{P}[S[\xi_2, \tau_1] \cap A = \emptyset, T \geq \tau_1 | \xi_2 < \infty, T \geq \xi_2] \leq C_4(m/R')^{1/2}. \tag{44}$$

We finally bound the probability that A is not hit between τ_1 and T , given that $T \geq \tau_1$. Consider $n' = \lambda^{-1/2+\varepsilon}$. We have $\mathbf{P}[T \leq \tau_{n'} | T \geq \tau_1] = O(\lambda(\log n')(n')^2) = o(\lambda^\varepsilon)$. Hence we are going to consider the probability that the walk avoids A up to time $\tau_{n'}$. Let

$$\tau_2 = \inf\{j \geq \tau_1 : S(j) \notin \mathbf{B}(2R')\}.$$

Define inductively the sequence of stopping times

$$\begin{aligned} \rho_1 &= \inf\{j : \tau_2 \leq j \leq \tau_{n'}, S(j) \in \mathbf{B}(R'/2)\} \\ \sigma_1 &= \inf\{j \geq \rho_1 : S(j) \notin \mathbf{B}(2R')\} \\ \rho_{i+1} &= \inf\{j : \rho_i \leq j \leq \tau_{n'}, S(j) \in \mathbf{B}(R'/2)\} \quad i \geq 1 \\ \sigma_{i+1} &= \inf\{j \geq \rho_{i+1} : S(j) \notin \mathbf{B}(2R')\}. \end{aligned}$$

Let $B_i := \{\sigma_i < \xi_A\}$, and let \mathcal{F}_i and \mathcal{G}_i , respectively, denote the σ -algebras generated by events up to time ρ_i and σ_i , respectively. Also let \mathcal{G}_0 be the σ -algebra generated by events up to time τ_2 . Since A contains a path from 0 to $\partial \mathbf{B}(R')$ we have $\mathbf{P}[B_i | \mathcal{F}_i] \leq c < 1$, $i \geq 1$. By considering the walk between σ_{i-1} and ρ_i , we also have

$$\mathbf{P}[\rho_i = \infty | \mathcal{G}_{i-1}] \leq \frac{C}{\log(n'/R')} \leq \frac{C'}{\log((\sqrt{\lambda}R')^{-1})}.$$

Write

$$\begin{aligned} F_i &= \cap_{\ell=1}^{i-1} \{\rho_\ell < \infty, B_\ell\} \cap \{\rho_i < \infty\} \\ G_i &= \cap_{\ell=1}^i \{\rho_\ell < \infty, B_\ell\}. \end{aligned}$$

Hence we deduce

$$\begin{aligned}
\mathbf{P}[S[\tau_2, \tau_{n'}] \cap A = \emptyset] &\leq \mathbf{P}\left[\bigcup_{i=1}^{\infty} \left[\left(\bigcap_{\ell=1}^{i-1} \{\rho_\ell < \infty, B_\ell\}\right) \cap \{\rho_i = \infty\}\right]\right] \\
&\leq \sum_{i=1}^{\infty} \left(\prod_{\ell=1}^{i-1} \mathbf{P}[\rho_\ell < \infty | G_{\ell-1}] \mathbf{P}[B_\ell | F_\ell] \right) \mathbf{P}[\rho_i = \infty | G_{i-1}] \\
&\leq \sum_{i=1}^{\infty} c^{i-1} \frac{C}{\log(n'/R')} \\
&\leq \frac{C'}{\log((\sqrt{\lambda} R')^{-1})}.
\end{aligned} \tag{45}$$

Using the Strong Markov property, we can combine the bounds (42), (43) and (45), and together with (41) we deduce (38).

Finally, letting $\lambda \rightarrow 0$ we obtain the second statement of the Proposition, as the bounds are uniform in λ . \square

We are ready to prove the analogue of Lemma 5 in the $d = 2$ case.

Proposition 3. *Assume $d = 2$. Let R, n and λ satisfy the relations as in Proposition 1. Let m, R' and λ satisfy the relations as in Proposition 2. There exists a coupling between $\text{LE}(S[0, T])$ and $\hat{S}[0, \infty)$, such that if $R = 4(R')^2$, then*

$$\begin{aligned}
\mathbf{P} \left[\begin{aligned} &\text{LE}(S[0, T]) \cap \mathbf{B}(m) = \hat{S}^\lambda[0, R] \cap \mathbf{B}(m); \\ &\hat{S}[0, \infty) \cap \mathbf{B}(m) = \hat{S}[0, R] \cap \mathbf{B}(m); \\ &\hat{S}^\lambda[0, R] = \hat{S}[0, R] \end{aligned} \right] \\
\geq 1 - O((m/R') \log(R'/m)) - O((\log R) R^{2-\beta}) \\
- O(\lambda n^2 R^{1+\beta} / \log[\lambda^{-1}]) - O\left(\frac{R^2}{n} \log \frac{n}{R}\right).
\end{aligned} \tag{46}$$

Proof. A self-avoiding walk of length $R = 4(R')^2$ necessarily visits $\partial \mathbf{B}(R')$. Consider the events in (36) and (37). On these events, $\text{LE}(S[0, T]) \cap \mathbf{B}(m) = \hat{S}^\lambda[0, R] \cap \mathbf{B}(m)$, and $\hat{S}[0, \infty) \cap \mathbf{B}(m) = \hat{S}[0, R] \cap \mathbf{B}(m)$. Due to Proposition 1, there exists a coupling between $\hat{S}[0, R]$ and $\hat{S}^\lambda[0, R]$ such that the two are identical with probability at least

$$1 - O((\log R) R^{2-\beta}) - O(\lambda n^2 R^{1+\beta} / \log[\lambda^{-1}]) - O\left(\frac{R^2}{n} \log \frac{n}{R}\right).$$

Hence we obtain the lemma. \square

From this point on, the proof of Theorem 1 is fairly similar to the $d = 3$ case. Using the notation of Section 3, the path $\pi_0^{(0)}$ is distributed as $\hat{S}[0, \infty)$, while the path $\pi_0^{(\gamma)}$ is distributed as $\text{LE}(S[0, T])$. Proposition 3 gives a coupling of the two paths. Let z_1, \dots, z_N be a list of all vertices in $\partial_{\text{ext}} B(k)$, and let z_{N+1}, \dots, z_{N+M} be a list of all

vertices in $B(k)$. Let S^i be independent random walks started at z_i , with geometric killing times T^i , $i = 1, \dots, N + M$. We use Wilson's algorithm with the walks S^i to construct both $\pi_{z_i}^{(\gamma)}$ and $\pi_{z_i}^{(0)}$. Write ξ_A^i , τ_N^i , etc. for hitting and exit times associated with S^i .

Let E_0 denote the event on the left hand side of (46). Assume this event occurs, and define $E_i = \{\xi_{\pi_0^{(0)}}^i < \tau_m^i, T^i > \tau_m^i\}$, $i = 1, \dots, N$. On the event $E_0 \cap (\cap_{i=1}^N E_i)$, the conditions (i)–(iii) of Section 3 hold for the paths $\pi_0^{(0)}, \pi_{z_1}^{(0)}, \dots, \pi_{z_N}^{(0)}$ and $\pi_0^{(\gamma)}, \pi_{z_1}^{(\gamma)}, \dots, \pi_{z_N}^{(\gamma)}$. The Beurling estimate and an argument similar to (20) gives

$$\mathbf{P}[E_i^c | E_0] \leq C(k/m)^{1/2} + C\lambda(\log m)m^2. \quad (47)$$

Note that the union of the paths $\pi_0^{(0)} \cup (\cup_{i=1}^N \pi_{z_i}^{(0)})$ separate $B(k)$ from $\partial \mathbf{B}(m)$. Hence the walks started at z_{N+1}, \dots, z_{N+M} necessarily hit the earlier paths before exiting $B(k)$. Let $F_i = \{T^i > \tau_m^i\}$, $i = N + 1, \dots, N + M$. On the event $E_0 \cap (\cap_{i=1}^N E_i) \cap (\cap_{i=N+1}^{N+M} F_i)$, the required coupling of paths is successful. Note that we have

$$\mathbf{P}[F_i^c | E_0 \cap (\cap_{i=1}^N E_i)] \leq C\lambda(\log k)k^2. \quad (48)$$

Combining (47) and (48) we get the following bound for the right hand side of (1):

$$\begin{aligned} & \mathbf{P}[E_0^c] + \sum_{i=1}^N \mathbf{P}[E_i^c | E_0] + \sum_{i=N+1}^{N+M} \mathbf{P}[F_i^c | E_0 \cap (\cap_{i=1}^N E_i)] \\ & \leq C(m/R') \log(R'/m) + C(\log R)R^{2-\beta} + C\lambda n^2 R^{1+\beta} / \log[\lambda^{-1}] \\ & \quad + C(R^2/n) \log(n/R) + C(k^{3/2}/m^{1/2}) + Ck\lambda(\log m)m^2 + C(\log k)k^4\lambda. \end{aligned}$$

Note that since $k < R$ and $m < n$, the term $k\lambda(\log m)m^2$ is of smaller order than the third term. Likewise, since $\beta > 2$, the term $(\log k)k^4\lambda$ is of smaller order than the third term. Omitting these terms we have the upper bound:

$$\begin{aligned} & C(m/R') \log(R'/m) + C(\log R)R^{2-\beta} + C\lambda n^2 R^{1+\beta} / \log[\lambda^{-1}] \\ & + C(R^2/n) \log(n/R) + C(k^{3/2}/m^{1/2}). \end{aligned} \quad (49)$$

We now choose the parameters. Setting $m^{-1/2} = m(R')^{-1}$ will make the first term of the same order as the last term (up to a logarithm). Hence we will choose $R' = m^{3/2}$, and hence $R = Cm^3$. We also set $R^2 n^{-1} = m^{-1/2}$, which makes the fourth term the same order as the last term (up to a logarithm). Therefore we take $n = m^{13/2}$. We set $R^{2-\beta} = R^2 n^{-1}$, which makes the second term the same order as the fourth. Hence we choose β determined by the relation: $R^\beta = n$. Finally, we set $\lambda n^2 R^{1+\beta} = \lambda n^3 R = m^{-1/2}$, which makes the third term the same order as the last one. This yields: $\lambda = m^{-23}$. Hence the optimal choice of m in terms of λ is $m = \lambda^{-1/23}$ ($\rho'' = 1/23$). This determines the other parameters as: $n = \lambda^{-13/46}$ ($\rho = 13/46$),

$R = 4\lambda^{-3/23}$ ($\rho' = 3/23$), $\beta = 13/2$, and $R' = \lambda^{-3/46}$. We need $2k < m = \lambda^{-1/23}$, and hence $\lambda \leq \lambda_0 := (2k)^{-23}$. With these choices the required relations between the parameters are satisfied: $\rho'' < \rho'/2$, $\rho' < \rho < 1/2$ and

$$\beta > 2; \quad 1 - 2\rho > (1 + \beta)\rho'; \quad 2\rho' < \rho; \quad (\rho'/2)4 < 1.$$

Hence Propositions 1, 2 and 3 apply, and the bound in (49) reduces to $Ck^{3/2}\lambda^{1/46-o(1)}$.

The better upper bound is obtained by setting each term equal to $k^{3/2}m^{-1/2}$. This yields: $R' = k^{-3/2}m^{3/2}$, $R = 4k^{-3}m^3$, $n = m^{13/2}k^{-15/2}$, $R^\beta = n$ and $\lambda = k^{27}m^{-23}$. Hence the optimal choice of the parameters in terms of λ and k is: $m = \lambda^{-1/23}k^{27/23}$, $R' = \lambda^{-3/46}k^6$, $R = 4\lambda^{-3/23}k^{12}$, $n = \lambda^{-13/46}k^{3/23}$. The restrictions on the parameters are satisfied as follows: $2k < m$, $16m < R'$, $R < n < \lambda^{-1/2}$ are automatic for $\lambda \leq \lambda_1$, with λ_1 independent of k . The condition $\beta > 2$ can be satisfied if

$$\frac{\frac{13}{46} \log(1/\lambda) + \frac{3}{23} \log k}{\log 4 + \frac{3}{23} \log(1/\lambda) + 12 \log k} > 2,$$

which holds if $\lambda < 16^{-46}k^{-46(24-\frac{3}{23})}$. So we can take $C_0 > 46(24-\frac{3}{23})$ and $c_0 = 16^{-46}$. The requirements $1 - 2\rho < (1 + \beta)\rho'$, $2\rho' < \rho$ and $(\rho'/2)4 < 1$ are automatically satisfied by the choice of the exponents. Hence for $\lambda \leq \lambda_0 := c_0k^{-C_0}$ we get the upper bound $Ck^{3/2}\lambda^{1/46-o(1)}k^{-27/46}$. This proves the theorem in the $d = 2$ case. \square

Acknowledgements. I thank Frank Redig and Ellen Saada for helpful comments.

References

- [1] Athreya, S.R. and Járai, A.A.: Infinite volume limit for the stationary distribution of Abelian sandpile models. *Commun. Math. Phys.* **249**, 197–213 (2004). An erratum for this paper appeared in *Commun. Math. Phys.* **264**, 843 (2006), with an electronic supplemental material.
- [2] Benjamini, I., Lyons, R., Peres, Y. and Schramm, O.: Uniform spanning forests. *Ann. Probab.* **29** 1–65 (2001).
- [3] Dhar D.: Self organized critical state of sandpile automaton models. *Phys. Rev. Letters* **64**, 1613–1616 (1990).
- [4] Gabrielov, A.: Abelian avalanches and Tutte polynomials. *Physica A* **195**, 253–274 (1993).
- [5] Holroyd, A.E., Levine, L., Mészáros, K., Peres, Y., Propp, J. and Wilson, D.B.: Chip-firing and rotor-routing on directed graphs. In: Sidoravicius, V. and Vares, M.E. (eds.) *In and out of equilibrium. 2*. *Progr. Probab.* Vol. **60**, Birkhäuser, Basel (2008) 331–364.

- [6] Járai, A.A., Redig, F.: Infinite volume limit of the Abelian sandpile model in dimensions $d \geq 3$.
- [7] Járai, A.A., Redig, F. and Saada, E.: Zero dissipation limit in the Abelian sandpile model. *Preprint*. <http://arxiv.org/abs/0906.3128v2> (2010).
- [8] Kesten, H.: Hitting probabilities of random walks on \mathbb{Z}^d . *Stochastic Process. Appl.* **25**, 165–184 (1987).
- [9] Lawler, G.F.: *Intersections of random walks*. Birkhäuser, Boston (1991).
- [10] Lawler, G.F.: Cut times for simple random walk. *Electron. J. Probab.* **1**, paper no. 13 (1996).
- [11] Lawler, G.F.: Loop-erased random walk. In: Bramson, M. and Durrett, R. (eds.) *Perplexing Problems in Probability*. Festschrift in Honour of Harry Kesten. pp. 197–217. Birkhäuser, Boston (1999).
- [12] Lawler, G.F. and Limic, V.: The Beurling estimate for a class of random walks. *Electron. J. Probab.* **9**, Paper No. 27, 846–861 (2004).
- [13] Majumdar, S.N. and Dhar, D.: Equivalence between the Abelian sandpile model and the $q \rightarrow 0$ limit of the Potts model. *Physica A* **185**, 129–145 (1992)
- [14] Masson, R.: The growth exponent for planar loop-erased random walk. *Electron. J. Probab.* **14**, 1012–1073 (2009).
- [15] Pemantle, R.: Choosing a spanning tree for the integer lattice uniformly. *Ann. Probab.* **19**, 1559–1574 (1991).
- [16] Redig, F.: Mathematical aspects of the Abelian Sandpile Model. Les Houches, Session LXXXIII 2005, A. Bovier, F. Dunlop, F. den Hollander, A. van Enter and J. Dalibard (eds.), Elsevier, pp. 657–728 (2006).
- [17] Wilson, D.B.: Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-Eighth ACM Symposium on the Theory of Computing*, 296–303; ACM, New York (1996).